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MATHEMATICS

TARGET IIT JEE

INDEFINITE INTEGRATION

THEORY AND EXERCISE BOOKLET

CONTENTS

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JEE Syllabus :

Integration as the inverse process of differentiation, indefinite integrals of standard functions, application of the Fundamental Theorem of Integral Calculus, integration by parts, integration by the methods of substitution and partial fractions

A. INTRODUCTION

Both parts of the Fundamental Theorem establish connections between antiderivatives and definite integrals. Part 1 says that if f is continuous, then $\int_a^x f(t)\,dt$ is an antiderivative of f. Part 2 says that $\int_a^b f(x)\,dx$ can be found by evaluating F(b) – F(a), where F is an antiderivative of f. We need a convenient notation for antiderivatives that makes them easy to work with. Because of the relation given by the Fundamental Theorem between antiderivatives and integrals, the notation $\int f(x)dx$ is traditionally used for antiderivative of f and is called and **indefinite integral**. Thus

$$\int f(x)dx = F(x) + C \qquad \text{means} \qquad F'(x) = f(x)$$

For example, we can write

$$\int x^2 dx = \frac{x^3}{3} + C \qquad \text{because} \qquad \frac{d}{dx}\left(\frac{x^3}{3} + C\right) = x^2$$

So we can regard an indefinite integral as representing an entire family of function (one antiderivative for each value of the constant C).

You should distinguish carefully between definite and indefinite integrals. A definite integral $\int_a^b f(x)\,dx$ is a number, whereas an indefinite integral $\int f(x)dx$ is family of functions. The connection between them is given by part 2 of the Fundamental Theorem. If f is continuous on [a, b] then

$$\int_a^b f(x)\,dx = \int f(x)dx\Big|_a^b$$

The effectiveness of the Fundamental Theorem depends on having a supply of antiderivatives of functions.

We adopt the convention that when a formula for a general indefinite integral is given, it is valid only on an interval. Thus, we write

$$\int \frac{1}{x^2}dx = -\frac{1}{x} + C$$

With the understanding that it is valid on the interval (0, ∞) or on the interval (–∞, 0). This true despite the fact that the general antiderivative of the function $f(x) = 1/x^2$, $x \neq 0$, is

$$F(x) \begin{cases} -\dfrac{1}{x} + C_1 & \text{if } x < 0 \\ -\dfrac{1}{x} + C_2 & \text{if } x > 0 \end{cases}$$

ELEMENTARY INTEGRALS

(i) $\int (ax+b)^n\,dx = \dfrac{(ax+b)^{n+1}}{a(n+1)} + c \quad n \neq -1$

(ii) $\int \dfrac{dx}{ax+b} = \dfrac{1}{a}\,\ell n\,|ax + b| + c$

(iii) $\int e^{ax+b}dx = \dfrac{1}{a}\,e^{ax+b} + c$

(iv) $\int a^{px+q}\,dx = \dfrac{a^{px+q}}{p\,\ell n\,a}\ (a > 0) + c$

(v) $\int \sin(ax+b)\,dx = -\dfrac{1}{a}\cos(ax + b) + c$

(vi) $\int \cos(ax+b)\,dx = \dfrac{1}{a}\sin(ax + b) + c$

(vii) $\int \tan(ax+b)\,dx = \frac{1}{a}\ \ell n \sec|ax+b| + c$

(viii) $\int \cot(ax+b)\,dx = \frac{1}{a}\ \ell n \sin|ax+b| + c$

(ix) $\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + c$

(x) $\int \operatorname{cosec}^2(ax+b)\,dx = -\frac{1}{a}\cot(ax+b) + c$

(xi) $\int \sec(ax+b).\tan(ax+b)\,dx = \frac{1}{a}\sec(ax+b) + c$

(xii) $\int \operatorname{cosec}(ax+b).\cot(ax+b)\,dx = -\frac{1}{a}\operatorname{cosec}(ax+b) + c$

(xiii) $\int \sec x\,dx = \ln|\sec x + \tan x| + c$ OR $\ln\tan\left|\frac{\pi}{4}+\frac{x}{2}\right| + c$ OR $-\ln|\sec x - \tan x| + c$

(xiv) $\int \operatorname{cosec} x\,dx = \ln|\operatorname{cosec} - \cot x| + c$ OR $\ln\left|\tan\frac{x}{2}\right| + c$ OR $-\ln|\operatorname{cosec}x + \cot x|$

B. INTEGRATION BY TRANSFORMATION

Ex.1 Integrate $\frac{1}{x^{3/4}} + \frac{1-x^4}{1-x}$ + sec x tan x.

Sol. Here I = $\int x^{-3/4}\,dx + \int (1+x+x^2+x^3)\,dx + \int \sec x \tan x\,dx$

[$\because$ $(1 - x^n)/(1 - x) = 1 + x + x^2 + x^3 + ... + x^{n-1}$]

$\therefore$ $I = 4x^{1/4} + x + (x^2/2) + (x^3/3) + (x^4/4) + \sec x + c$.

Ex.2 Integrate (1 + sin x)/(1 – cos x).

Sol. Here $I = \int \frac{1+\sin x}{1-\cos x}\,dx = \int \frac{1+2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\,dx = \frac{1}{2}\int \operatorname{cosec}^2\frac{x}{2}\,dx + \int \cot\frac{x}{2}\,dx = -\cot\frac{x}{2} + 2\log\left(\sin\frac{x}{2}\right) + c$

Ex.3 Evaluate $\int \frac{dx}{\tan x + \cot x + \sec x + \operatorname{cosec} x}$

Sol. Here, $I = \int \frac{dx}{\tan x + \cot x + \sec x + \operatorname{cosec} x} = \int \frac{(\sin x \cos x)\,dx}{1+\sin x + \cos x} = \int \frac{\sin x}{1+\tan x + \sec x}\,dx$

Multiplying and dividing by (1 + tan x – sec x), we get

$$= \int \frac{\sin x(1+\tan x - \sec x)}{(1+\tan x)^2 - \sec^2 x}\,dx = \int \frac{\sin x(1+\tan x - \sec x)}{2\tan x}\,dx$$

$$= \frac{1}{2}\int \cos x(1+\tan x - \sec x)\,dx = \frac{1}{2}\int (\cos x + \sin x - 1)\,dx = \frac{1}{2}\int (\sin x - \cos x - x) + c$$

Ex.4 Integrate $\frac{5\cos^3 x + 2\sin^3 x}{2\sin^2 x\cos^2 x}$ + $\sqrt{(1+\sin 2x)}$ + $\frac{1+2\sin x}{\cos^2 x}$ + $\frac{1-\cos 2x}{1+\cos 2x}$

Sol. The given expression may be written as

$$\frac{5\cos^3 x + 2\sin^3 x}{2\sin^2 x\cos^2 x} + \sqrt{(\cos^2 x + \sin^2 x + 2\sin x\cos x)} + \frac{1}{\cos^2 x} + \frac{2\sin x}{\cos^2 x} + \frac{2\sin^2 x}{2\cos^2 x}$$

$= \frac{5}{2}$ cosec x cot x + sec x tan x + cos x + sin x + $\sec^2$ x + 2 sec x tan x + 2 ($\sec^2$ x – 1)

$= \frac{5}{2}$ cosec x cot x + 3 sec x tan x + cos x + sin x + 3 $\sec^2$ x – 2.

Now integrating, we get

$$I = \frac{5}{2}\int \text{cosec } x \cot x\, dx + 3\int \sec x \tan x\, dx + \int \cos x\, dx + \int \sin x\, dx + 3\int \sec^2 x\, dx - 2\int dx$$

$= -\frac{5}{2}$ cosec x + 3 sec x + sin x – cos x + 3 tan x – 2x + c.

Ex.5 Integrate $\frac{\sec x}{\sqrt{3+\tan x}}$

Sol. We have $\int \frac{\sec x\, dx}{\sqrt{3+\tan x}} = \int \frac{\sec x\, dx}{\sqrt{3+(\sin x/\cos x)}} = \int \frac{dx}{\sqrt{3}+\cos x + \sin x} = \int \frac{dx}{2\left[(\sqrt{3/2})\cos x + \frac{1}{2}\sin x\right]}$

$$\frac{1}{2}\int \frac{dx}{\sin\left(x+\frac{1}{3}\pi\right)} = \frac{1}{2}\int \text{cosec}\left(x+\frac{1}{3}\pi\right)dx = \frac{1}{2}\log\left|\tan\left[\frac{1}{2}x + (\pi/6)\right]\right| + c.$$

CONSIDER INTEGRALS OF THE TYPES

$\int \cos ax \cos bx\, dx$, $\int \cos ax \cos bx\, dx$, $\int \cos ax \cos bx\, dx$, in which $a \neq b$.

We can use these addition formulae to change products to sums or differences, and the later can be integrated easily.

Ex.6 Integrate $\int \sin 8x \sin 3x\, dx$

Sol. sin 8x sin 3x = $\frac{1}{2}$ (cos 5x – cos 11x),

and so $\int \sin 8x \sin 3x\, dx\, x = \frac{1}{2}\int (\cos 5x - \cos 11x)\, dx = \frac{1}{10}$ sin 5x – $\frac{1}{22}$ sin 11x + c,

We next consider integrals of the type $\int \cos^m x \sin^n x\, dx$, in which at least one of the exponents m and n is an odd positive integer (the other exponent need only be a real number). Suppose that m = 2k + 1, where k is a non-negative integer. Then $\cos^m x \sin^n x = \cos^{2k+1} x \sin^n x = (\cos^2 x)^k \sin^n x \cos x$.

Using the identity $\cos^2 x = 1 - \sin^2 x$, we obtain $\int \cos^m . x \sin^n x\, dx = \int (1-\sin^2 x)^k \sin^n x \cos x\, dx$.

The factor $(1 - \sin^2 x)^k$ can be expanded by the Binomial Theorem, and the result is that $\int \cos^m x \sin^n x\, dx$ can be written as a sum of constant multiples of integrals of the form $\int \sin^q x \cos x\, dx$. Since

$$\int \sin^q x \cos x\, dx = \begin{cases} \dfrac{1}{q+1}\sin^{q+1} x + c & \text{if } q \neq -1, \\ \ell n\,|\sin x| + c & \text{if } q = -1, \end{cases}$$

If follows that $\int \cos^m x \sin^n x\, dx$ can be readily evaluated, An entirely analogous argument follows if the exponent n is an odd positive integer.

Ex.7 Integrate **(a)** $\int \cos^3 4x\, dx$, **(b)** $\int \sin^5 x \cos^4 x\, dx$

Sol. In **(a)** illustrates that the method just described is applicable to odd positive integer powers of the sine or cosine (i.e., either m or n may be zero). We obtain

$$\int \cos^3 4x\, dx = \int \cos^2 4x \cos 4x\, dx = \int (1 - \sin^2 4x)\cos 4x\, dx$$

$$= \int \cos 4x\, dx - \int \sin^2 4x \cos 4x\, dx = \frac{1}{4}\sin 4x - \frac{1}{12}\sin^3 4x + c.$$

In **(b)** it is the exponent of the sine which is an odd positive integer.

$$\int \sin^5 x \cos^4 x\, dx = \int (\sin^2 x)^2 \cos^4 x \sin x\, dx$$

$$= \int (1 - \cos^2 x)^2 \cos^4 x \sin x\, dx \qquad = \int (1 - 2\cos^2 x + \cos^4 x)\cos^4 x \sin x\, dx$$

$$= \int \cos^4 x \sin x\, dx - 2\int \cos^6 x \sin x\, dx + \int \cos^8 x \sin x\, dx = \frac{1}{5}\cos^5 x + \frac{2}{7}\cos^7 x - \frac{1}{9}\cos^9 x + c.$$

The third type of integral we consider consists of those of the form $\int \cos^m x \sin^n x\, dx$,

in which both m and n are even non-negative integers. These function are not so simple to integrate as those containing an odd power. We first consider the special case in which either m = 0 or n = 0. The simplest non-trivial examples are the two integrals $\int \cos^2 x\, dx$ and $\int \sin^2 x\, dx$, which can be integrated by means of the identities $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$, $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

Evaluation of the two integrals is now a simple matter. We get

$$\int \cos^2 x\, dx = \frac{1}{2}\int (1 + \cos 2x)\, dx = \frac{x}{2} + \frac{1}{4}\sin 2x + c,$$

$$\int \sin^2 x\, dx = \frac{1}{2}\int (1 - \cos 2x)\, dx = \frac{x}{2} + \frac{1}{4}\sin 2x + c.$$

Going on to the higher powers, consider the integral $\int \cos^{2i} x\, dx$, where i is an arbitrary positive integer.

We write $\cos^{2i} x = (\cos^2 x)^i = \left[\frac{1}{2}(1 + \cos 2x)\right]^i = \frac{1}{2^i}(1 + \cos 2x)^i$

We expand using binomial theorem and integrate the terms using previous methods.

C. INTEGRATION BY SUBSTITUTION

Let g be a function whose range is an interval I, and let f be a function that is continuous on I. If g is differentiable on its domain and F is an antiderivative of f on I, then $\int f(g(x))g'(x)\,dx$ = F(g(x)) + C.

If u = g(x), then du = g'(x) and $\int f(u)\,du = F(u) + C$.

GUIDELINES FOR MAKING A CHANGE OF VARIABLE

1. Choose a substitution u = g(x). Usually, it is best to choose the inner part of a composite function, such as a quantity raised to a power.
2. Compute du = g'(x) dx.
3. Rewrite the integral in terms of the variable u.
4. Evaluate the resulting integral in terms of u.
5. Replace u by g(x) to obtain an antiderivative in terms of x.

THE GENERAL POSER RULE FOR INTEGRATION

If g is a differentiable function of x, then $\int [g(x)]^n\, g'(x)\,dx = \dfrac{[g(x)]^{n+1}}{n+1}$ + C, $n \neq -1$

RATIONALIZING SUBSTITUTIONS

Some irrational functions can be changed into rational functions by means of appropriate substitutions. In particular, when an integrand contains an expression of the form $\sqrt[n]{g(x)}$, then the substitution

u = $\sqrt[n]{g(x)}$ may be effective.

SOME STANDARD SUBSTITUTIONS

(i) $\int \dfrac{dx}{x(x^n+1)}$ $n \in N$ Take x^n common & put $1 + x^{-n} = t$.

(ii) $\int \dfrac{dx}{x^2(x^n+1)^{(n-1)/n}}$ $n \in N$, take x^n common & put $1 + x^{-n} = t^n$

(iii) $\int \dfrac{dx}{x^n(1+x^n)^{1/n}}$ take x^n common as x and put $1 + x^{-n} = t$.

(iv) $\int \sqrt{\dfrac{x-\alpha}{\beta-x}}\,dx$ or $\int \sqrt{(x-\alpha)(\beta-x)}$; put $x = \alpha\cos^2\theta + \beta\sin^2\theta$

(v) $\int \sqrt{\dfrac{x-\alpha}{x-\beta}}\,dx$ or $\int \sqrt{(x-\alpha)(x-\beta)}$; put $x = \alpha\sec^2\theta - \beta\tan^2\theta$

Ex.8 Evaluate $\int (x^2+1)^2\,(2x)\,dx$.

Sol. Letting $g(x) = x^2 + 1$, we obtain $g'(x) = 2x$ and $f(g(x)) = [g(x)]^2$.
From this, we can recognize that the integrand and follows the f(g(x)) g'(x) pattern. Thus, we can write $\int \overbrace{(x^2+1)^2}^{[g(x)]^2}\,\overbrace{(2x)}^{g'(x)}\,dx = \dfrac{1}{3}(x^2+1)^3 + C$.

Ex.9 Evaluate $\int \frac{-4x}{(1-2x^2)^2}\,dx$

Sol. $\int \underbrace{(1-2x^2)^{-2}}_{u^{-2}}\underbrace{(-4x)\,dx}_{du} = \frac{\overbrace{(1-2x^2)^{-1}}^{u^{-1}(-1)}}{-1} + C$

Ex.10 Evaluate $\int x^3 \cos(x^4+2)\,dx$.

Sol. Let $u = x^4 + 2 \Rightarrow du = 4x^3\,dx$

$\int x^3 \cos(x^4+2)\,dx = \int \cos u.\frac{1}{4}du = \frac{1}{4}\int \cos u\,du = \frac{1}{4}\sin u + C = \frac{1}{4}\sin(x^4+2) + C$

Ex.11 Evaluate $\int \frac{x^2dx}{(x^3-2)^5}$.

Sol. Let $u = x^3 - 2$. Then $du = 3x^2\,dx$. so by substitution :

$\int \frac{x^2dx}{(x^3-2)^5} = \int \frac{du/3}{u^5} = \frac{1}{5}\int u^{-5}du = \frac{1}{3}\frac{u^{-4}}{-4} + C = -\frac{1}{12}(x^3-2)^{-4} + C.$

Ex.12 Evaluate $\int \frac{\sqrt{x+4}}{x}\,dx$

Sol. Let $u = \sqrt{x+4}$. Then $u^2 = x + 4$, so $x = u^2 - 4$ and $dx = 2u\,du$.

Therefore $\int \frac{\sqrt{x+4}}{x}\,dx = \int \frac{u}{u^2-4}\,2u\,du = 2\int \frac{u^2}{u^2-4}\,du = 2\int\left(1+\frac{4}{u^2-4}\right)du$

$= 2\int du + 8\int \frac{du}{u^2-4} = 2u + 8\,.\,\frac{1}{2\cdot 2}\,\ell n\left|\frac{u-2}{u+2}\right| + C = 2\sqrt{x+4} + 2\,\ell n\left|\frac{\sqrt{x+4}-2}{\sqrt{x+4}+2}\right| + C$

Ex.13 Evaluate $\int \frac{dx}{1+e^x}$

Sol. Rewrite the integrand as follows : $\frac{1}{1+e^x} = \frac{e^{-x}}{e^{-x}}\left(\frac{1}{1+e^x}\right) = \frac{e^{-x}}{e^{-x}+1}$ $(u = e^{-x} + 1;\ du = -e^{-x}\,dx)$

$\int \frac{dx}{1+e^x} \int \frac{e^{-x}dx}{e^{-x}+1} = \int \frac{-du}{u} = -\ell n|u| + C = -\ell n(e^{-x}+1) + c$ $(\because\ e^{-x} + 1 > 0)$

Ex.14 Evaluate $\int \sec x\,dx$

Sol. Multiply the integrand sec x by sec x + tan x and divide by the same quantity :

$\int \sec x\,dx = \int \frac{\sec(\sec x + \tan x)}{\sec x + \tan x}\,dx = \int \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}\,dx$

Now put $u = \sec x + \tan x \Rightarrow du = (\sec x \tan x + \sec^2 x)\,dx$

we find $\int \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}\,dx = \int \frac{du}{u} = \ell n|u| + C = \ell n|\sec x + \tan x| + C$

Ex.15 Evaluate $\int \cos x \sqrt{(4-\sin^2 x)}\, dx$

Sol. Put sin x = t so that cos x dx = dt. Then the given integral = $\int \sqrt{(4-t^2)}\, dt = \int \sqrt{(2^2-t^2)}\, dt$

$= \frac{1}{2} t \sqrt{(2^2-t^2)} + \frac{2^2}{2} \sin^{-1}(t/2) + c = \frac{1}{2} \sin x \,.\, \sqrt{(4-\sin^2 x)} + 2 \sin^{-1}(1/2 \sin x) + c$

Ex.16 Integrate **(i)** $\frac{e^x - e^{-x}}{e^x + e^{-x}}$, **(ii)** $\frac{10x^9 + 10^x . \log_e 10}{10^x + x^{10}}$

Sol. **(i)** Let I = $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$. Now putting $e^x + e^{-x} = t$, so that $(e^x - e^{-x})\, dx = dt$,

we have I = $\int (1/t)\, dt$ = log t = log $(e^x + e^{-x})$.

(ii) Here I = $\int \frac{10x^9 + 10^x . \log_e 10}{10^x + x^{10}} dx$. Now putting $10^x + x^{10} = t$, and $(10^x \log_e 10 + 10x^9)\, dx = dt$,

we have I = $\int (1/t)\, dt = \log t = \log(10^x + x^{10}) + c$

Ex.17 Integrate **(i)** $\frac{1}{x \cos^2(1+\log x)}$, **(ii)** $\frac{1}{x(1+\log x)^m}$

Sol. **(i)** Here I = $\int dx / \{x \cos^2(1+\log x)\}$. Putting 1 + log x = t, so that (1/x) dx = dt, we have

$$I = \int dt/\cos^2 t = \int \sec^2 t\, dt = \tan t = \tan(1 + \log x) + c.$$

(ii) Here I = $\int dx / \{x(1+\log x)^m\}$. Putting 1 + log x = t, so that (1/x)dx = dt, we have

$$I = \int \frac{dt}{t^m} = \frac{t^{-m+1}}{-m+1} = \frac{(1+\log x)^{-m+1}}{(1-m)} = \frac{1}{(1-m)} (1 + \log x)^{1-m} + c.$$

Ex.18 Integrate **(i)** $\frac{\cot x}{\log(\sin x)}$ **(ii)** $\frac{\tan x}{(\log(\sec x)}$

Sol. **(i)** Here $\frac{d}{dx}(\log \sin x) = \frac{1}{\sin x} \cos x = \cot x$.

$\therefore$ I = $\int \frac{\cot x\, dx}{\log \sin x} = \int \frac{\cot x}{t} \times \frac{dt}{\cot x}$ = log |log (sin x)|+ c

(ii) We have $\int \frac{\tan x\, dx}{\log \sec x}$ = log |log sec x| + c

Ex.19 Integrate $\sqrt{(1+\sin x)}$

Sol. We have I = $\int \sqrt{(1+\sin x)}\,dx = \int \sqrt{\left\{1-\cos\left(\frac{\pi}{2}+x\right)\right\}}\,dx = \int \sqrt{\left[2\sin^2\left(\frac{\pi}{4}+\frac{x}{2}\right)\right]}\,dx$

Now put $\frac{x}{2}+\frac{\pi}{4} = t \Rightarrow \frac{1}{2}dx = dt$ or $dx = 2\,dt$, we have $I = \int \sqrt{2\sin^2 t}\,(2dt) = -2\sqrt{2}\cos\left(\frac{x}{2}+\frac{\pi}{4}\right)+c$

Ex.20 Integrate $\cos^5 x$.

Sol. $\int \cos^5 x\,dx = \int \cos^4 x\cos x\,dx = \int (1-\sin^2 x)^2\cos x\,dx = \int (1-t^2)^2\,dt$, [put $\sin x = t \Rightarrow \cos x\,dx = dt$]

$$= \int (1-2t^2+t^4)\,dt = t - \frac{2}{3}t^3 + \frac{1}{5}t^5 + c = \frac{\sin^5 x}{5} - \frac{2}{3}\sin^3 x + \sin x + c$$

Ex.21 Evaluate $\int \frac{\cos^5 x}{\sin^2 x}dx$

Sol. Let $I = \int \frac{\cos^5 x}{\sin^2 x}dx = \int \frac{\cos^4 x}{\sin^2 x}\cos x\,dx = \int \frac{(1-\sin^2 x)^2}{\sin^2 x}\cos x\,dx$ [put $\sin x = t \Rightarrow \cos x\,dx = dt$]

then $I = \int \frac{(1-t^2)^2}{t^2} = dt \int \frac{1-2t^2+t^4}{t^2}dt = \int \left[\frac{1}{t^2}-2+t^2\right]dt = -\frac{1}{t} - 2t + \frac{t^3}{3}$

$$= -\frac{1}{\sin x} - 2\sin x + \frac{\sin^3 x}{3} = -\operatorname{cosec} x - 2\sin x + \frac{1}{3}\sin^3 x + c$$

Ex.22 Integrate $1/(\sin^3 x\cos^5 x)$.

Sol. Here the integrand is $\sin^{-3} x\cos^{-5} x$. It is of type $\sin^m x\cos^n x$,
where $m + n = -3 -5 = -8$ i.e., –ve even integer

$$\therefore\ I = \int \frac{dx}{\sin^3 x\cos^5 x} = \int \frac{dx}{(\sin^3 x/\cos^3 x)\cos^3 x\cos^5 x} = \int \frac{\sec^8 x\,dx}{\tan^3 x} = \int \frac{\sec^6 x.\sec^2 x\,dx}{(\tan^3 x)} = \int \frac{(1+\tan^2 x)^3\sec^2 x\,dx}{\tan^3 x}$$

Now put $\tan x = t$ so that $\sec^2 x\,dx = dt$

$$\therefore\ I = \int \frac{(1+t^2)^3dt}{t^3} = \int\left(\frac{1}{t^3}+\frac{3}{t}+3t+t^2\right)dt = -\frac{1}{2t^2} + 3\log t + \frac{3}{2}\tan^2 x + \frac{1}{3}\tan^3 x$$

Ex.23 Integrate $1/\sqrt{(\cos^3 x\sin^5 x)}$

Sol. Here the integrand is of the type $\cos^m x\sin^n x$. We have $m = -3/2$, $n = -5/2$, $m + n = -4$ i.e., and even negative integer. $\therefore$ $\int \frac{dx}{\sqrt{(\cos^3 x\sin^5 x)}} = \int \frac{dx}{\cos^{3/2} x\sin^{5/2} x} = \int \frac{dx}{\cos^{3/2} x(\sin^{5/2} x/\cos^{5/2} x)\cos^{5/2} x}$

$$= \int \frac{dx}{\cos^4 x\tan^{5/2} x} = \int \frac{\sec^4 x}{\tan^{5/2} x}dx = \int \frac{\sec^2 x}{\tan^{5/2} x}\sec^2 x\,dx$$

$$= \int \frac{(1+\tan^2 x)}{\tan^{5/2} x}\sec^2 x\,dx = \int \frac{(1+t^2)}{t^{5/2}}dt, \text{ putting } \tan x = t \text{ and } \sec^2 x\,dx = dt$$

$$= \int (t^{-5/2}+t^{-1/2})dt = -\frac{2}{3}t^{-3/2} + 2t^{1/2} = -\frac{2}{3}(\tan x)^{-3/2} + 2(\tan x)^{1/2} = 2\sqrt{(\tan x)} - \frac{2}{3}(\tan x)^{-3/2} + c$$

Ex.24 Evaluate $\int \frac{dx}{\sqrt{\sin(x+\alpha)\cos^3(x-\beta)}}$

Sol. Put $x - \beta = y \Rightarrow dx = dy$

Given integral $I = \int \frac{dy}{\sqrt{\cos^3 y \sin(y+\beta+\alpha)}} \Rightarrow I = \int \frac{dy}{\sqrt{\cos^3 y \sin(y+\theta)}}$ $(\theta = \alpha + \beta)$

$= \int \frac{dy}{\sqrt{\cos^3 y(\sin y \cos\theta + \cos y \sin\theta)}} = \int \frac{dy}{\sqrt{\cos^4 y(\cos\theta \tan y + \sin\theta)}}$

$= \int \frac{\sec^2 y dy}{\sqrt{(\cos\theta \tan y + \sin\theta)}}$ Now put $\sin\theta + \cos\theta \tan y = z^2 \Rightarrow \cos\theta \sec^2 y\, dy = 2z\, dz$

$\Rightarrow \quad I = \int \frac{2z \sec\theta dz}{z} \quad \Rightarrow \quad 2\sec\theta \sqrt{\frac{\sin(y+\theta)}{\cos y}} + c = 2\sec(\alpha+\beta)\sqrt{\frac{\sin(x+\alpha)}{\cos(x-\beta)}} + c$

Ex.25 Evaluate $\int \frac{5x^4 + 4x^5}{(x^5+x+1)^2} dx$

Sol. $I = \int \frac{5x^4+4x^5}{(x^5+x+1)^2} dx = \int \frac{x^4(5+4x)dx}{x^{10}\left(1+\frac{1}{x^4}+\frac{1}{x^5}\right)^2} = \int \frac{5/x^6 + 4/x^5}{\left(1+\frac{1}{x^4}+\frac{1}{x^5}\right)^2} dx,$

put $1 + \frac{1}{x^4} + \frac{1}{x^5} = t \Rightarrow \left(-\frac{5}{x^5} - \frac{6}{x^6}\right) dx = dt = \int -\frac{dt}{t^2} = \frac{1}{t} + c = \frac{1}{1+\frac{1}{x^4}+\frac{1}{x^5}} + c = \frac{x^5}{x^5+x+1} + c$

D. RATIONALIZATION BY TRIGONOMETRIC SUBSTITUTION

Consider the integral $\int \sqrt{a^2 - x^2}\, dx$

If we change the variable from x to θ by the substitution $x = a \sin\theta$, then the identity $1 - \sin^2\theta = \cos^2\theta$ allows us to get rid of the roots sign because

$\int \sqrt{a^2-x^2} = \sqrt{a^2 - a^2\sin^2\theta} = \sqrt{a^2(1-\sin^2\theta)} = \sqrt{a^2\cos^2\theta} = a|\cos\theta|$

Notice the difference between the substitution $u = a^2 - x^2$ (in which the new variable is a function of the old one) are the substitution $x = a \sin\theta$ (the old variable is a function of the new one).
In general we can make a substitution of the form $x = g(t)$ by using the Substitution Rule in reverse. To make our calculations simpler, we assume that g has an inverse function; that is, g is one-to-one.

$\int f(x)dx = \int f(g(t))g'(t)dt$

This kind of substitution is called inverse substitution.
We can make the inverse substitution $x = a \sin\theta$ provided that it defines a one-to-one function. This can be accomplished by restricting θ to lie in the interval $[-\pi/2, \pi/2]$.

SOME STANDARD INTEGRALS

(i) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + c$

(ii) $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

(iii) $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + c$

(iv) $\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln\,[x + \sqrt{x^2 + a^2}]$

(v) $\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln\,[x + \sqrt{x^2 - a^2}]$

(vi) $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + c$

(vii) $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$

(viii) $\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

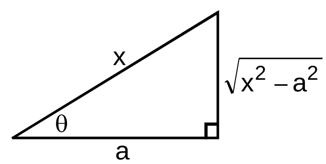
(ix) $\int \sqrt{x^2 + a^2}\,dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\ln\left|x + \sqrt{x^2 + a^2}\right| + c$

(x) $\int \sqrt{x^2 - a^2}\,dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\ln\left|x + \sqrt{x^2 - a^2}\right| + c$

Ex.26 Evaluate $\int \frac{dx}{\sqrt{x^2 - a^2}}$, where $a > 0$

Sol. We let $x = a\sec\theta$, where $0 < \theta < \pi/2$ or $\pi < \theta < 3\pi/2$. Then $dx = a\sec\theta\tan\theta\,d\theta$ and

$\sqrt{x^2 - a^2} = \sqrt{a^2(\sec^2\theta - 1)} = \sqrt{a^2\tan^2\theta} = a\,|\tan\theta| = a\tan\theta$



Therefore $\int \frac{dx}{\sqrt{x^2 - a^2}} = \int \frac{a\sec\theta\tan\theta}{a\tan\theta}\,d\theta = \int \sec\theta\,d\theta = \ln\,|\sec\theta + \tan\theta| + c$

The triangle in figure gives $\tan\theta = \frac{\sqrt{x^2 - a^2}}{a}$, so we have $\frac{dx}{\sqrt{x^2 - a^2}} = \ln\left|\frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a}\right| + c$

$= \ln\,|x + \sqrt{x^2 - a^2}\,| - \ln a + C = \ln\,|x + \sqrt{x^2 - a^2}\,| + C$, $\sec\theta = \frac{x}{a}$

Ex.27 Integrate $1/(2x^2 + x - 1)$.

Sol. We have $\int \frac{dx}{(2x^2 + x - 1)} = \frac{1}{2}\int \frac{dx}{\left(x^2 + \frac{x}{2} - \frac{1}{2}\right)}$

$$= \frac{1}{2}\int \frac{dx}{\left(x + \frac{1}{4}\right)^2 - \frac{1}{2} - \frac{1}{16}} = \frac{1}{2}\int \frac{dx}{\left(x + \frac{1}{4}\right)^2 - \frac{9}{16}} = \frac{1}{2}.\frac{1}{2}\,\frac{4}{3}\log\frac{x + \frac{1}{4} - \frac{3}{4}}{x + \frac{1}{4} + \frac{3}{4}} + C$$

$$= \frac{1}{3}\log\left|\frac{2x-1}{2(x+1)}\right| + c = \frac{1}{3}\log\left|\frac{2x-1}{x+1}\right| - \frac{1}{3}\log 2 + c = \frac{1}{3}\log\,(2x - 1)/(x - 1)| + C_1$$

Ex.28 Integrate $(3x + 1) / (2x^2 - 2x + 3)$.

Sol. Here $(d/dx)\,(2x^2 - 2x + 3) = 4x - 2$.

$$\therefore \quad I = \int \frac{3x+1}{2x^2-2x+3}\,dx = \int \frac{\frac{3}{4}(4x-2)+1+\frac{3}{2}}{(2x^2-2x+3)}\,dx$$

$$= \frac{3}{4}\int \frac{4x-2}{2x^2-2x+3}\,dx + \frac{5}{2}\int \frac{1}{2x^2-2x+3}\,dx = \frac{3}{4}\log(2x^2-2x+3) + \frac{5}{2.2}\int \frac{dx}{x^2-x+(3/2)}$$

$$= \frac{3}{4}\log(2x^2-2x+3) + \frac{5}{4}\int \frac{dx}{\left(x-\frac{1}{2}\right)^2+\left(\frac{3}{2}\right)-\left(\frac{1}{4}\right)} = \frac{3}{4}\log(2x^2-2x+3) + \frac{5}{4}\int \frac{dx}{\left(x-\frac{1}{2}\right)^2+(\sqrt{5/2})^2}$$

$$= \frac{3}{4}\log(2x^2-2x+3) + \frac{5}{4}\,\frac{1}{(\sqrt{5/2})}\left(\tan^{-1}\left\{\frac{x-\frac{1}{2}}{(\sqrt{5/2})}\right\}\right) + c$$

$$= \frac{3}{4}\log(2x^2-2x+3) + \frac{\sqrt{5}}{2\sqrt{2}}\tan^{-1}\left(\frac{2x-1}{\sqrt{10}}\right) + c$$

Ex.29 Integrate $1/\sqrt{(4+3x-2x^2)}$.

Sol. We have $\displaystyle\int \frac{dx}{\sqrt{(4+3x-2x^2)}} = \frac{1}{\sqrt{2}}\int \frac{dx}{\left\{2+\frac{9}{16}-\left(x^2-\frac{3}{2}x+\frac{9}{16}\right)\right\}} = \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{\{(41/16)-(x-3/4)^2\}}}$

$$= \frac{1}{\sqrt{2}}\sin^{-1}\left\{\frac{x-\frac{3}{4}}{(\sqrt{41/4})}\right\} + c = \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{4x-3}{41}\right) + c$$

Ex.30 Evaluate $\int \sqrt{(x-1)(2-x)}\,dx$.

Sol. We have $\int \sqrt{(x-1)(2-x)}\,dx = \int \sqrt{(-x^2+3x-2)}\,dx$

$$= \int \sqrt{\left\{-2-\left(x-\frac{3}{2}\right)^2+\frac{9}{4}\right\}}\,dx \int \sqrt{\left\{\frac{1}{4}-\left(x-\frac{3}{2}\right)^2\right\}}\,dx, \text{ [form } \int \sqrt{(a^2-x^2)}\,dx\text{]}$$

$$= \frac{1}{2}\left(x-\frac{3}{2}\right)\sqrt{\left\{\frac{1}{4}-\left(x-\frac{3}{2}\right)^2\right\}} + \frac{1}{2}\cdot\frac{1}{4}\sin^{-1}\left\{\left(x-\frac{3}{2}\right)/(1/2)\right\} + c$$

$$= \frac{1}{4}(2x-3)\sqrt{(3x-x^2-2)} + \frac{1}{8}\sin^{-1}(2x-3) + c$$

Ex.31 Integrate $\int \frac{(x^3+3)dx}{\sqrt{(x^2+1)}}$

Sol. We have $\int \frac{(x^3+3)dx}{\sqrt{(x^2+1)}} = \int \frac{x(x^2+1)-x+3}{\sqrt{(x^2+1)}} dx = \int \frac{x(x^2+1)}{\sqrt{(x^2+1)}} dx - \int \frac{xdx}{\sqrt{(x^2+1)}} + 3\int \frac{dx}{\sqrt{(x^2+1)}}$

$= \frac{1}{2}\int (2x)\sqrt{(x^2+1)}\, dx - \frac{1}{2}\int \frac{2x\,dx}{\sqrt{(x^2+1)}} + 3\int \frac{dx}{\sqrt{(x^2+1)}}$

$= \frac{1}{2}\left[\frac{2}{3}(x^2+1)^{3/2}\right] - \frac{1}{2}\,[2\sqrt{(x^2+1)}\,] + 3\,\ell n\,(x + \sqrt{(x^2+1)}\,) + c$

$= \frac{1}{3}\,(x^2+1)^{2/3} - \sqrt{(x^2+1)} + 3\,\ell n\,(x + \sqrt{(x^2+1)}\,) + c$

Ex.32 Integrate $x^2/(x^4 + x^2 + 1)$

Sol. Let $I = \int \frac{x^2}{x^4+x^2+1}\,dx, = \int \frac{1}{x^2+1+\frac{1}{x^2}}\,dx$, dividing the numerator and the denominator both by x^2.

Now the denominator $x^2 + 1 + \frac{1}{x^2}$ can be written either as $\left(x-\frac{1}{x}\right)^2 + 1$ or as $\left(x+\frac{1}{x}\right)^2 - 1$. The diff.

coeff. of $x - \frac{1}{x}$ is $1 + \frac{1}{x^2}$ and that of $x + \frac{1}{x}$ is $1 - \frac{1}{x^2}$. So we write

$$I = \frac{1}{2}\int \frac{(1+1/x^2)+(1-1/x^2)}{x^2+1+(1/x^2)}\,dx = \frac{1}{2}\int \frac{(1+1/x^2)dx}{(x-1/x)^2+3} + \frac{1}{2}\int \frac{(1-1/x^2)dx}{(x+1/x)^2-1}$$

In the first integral put $x - \frac{1}{x} = t$ so that $\left(1+\frac{1}{x^2}\right)dx = dt$, and in the second integral put

$x + \frac{1}{x} = z$ so that $\left(1-\frac{1}{x^2}\right)dx = dz$.

$$\therefore\quad I = \frac{1}{2}\int \frac{dt}{t^2+(\sqrt{3})^2} + \frac{1}{2}\int \frac{dz}{z^2-1} = \frac{1}{2\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}} + \frac{1}{2}\,\frac{1}{2\times 1}\log\frac{z-1}{z+1} + c$$

$$\frac{1}{2\sqrt{3}}\tan^{-1}\left\{\frac{(x-1/x)}{\sqrt{3}}\right\} + \frac{1}{4}\log\frac{(x+1/x)-1}{(x+1/x)+1} + c = \frac{1}{2\sqrt{3}}\tan^{-1}\left\{\frac{x^2-1}{(\sqrt{3})x}\right\} + \frac{1}{4}\log\frac{x^2-x+1}{x^2+x+1} + c$$

Ex.33 Evaluate $\int \frac{(1+x^2)dx}{(1-x^2)\sqrt{1+x^2+x^4}}$

Sol. Let, I = $\int \frac{(1+x^2)dx}{(1-x^2)\sqrt{1+x^2+x^4}}$ = $\int \frac{x^2\left(1+\frac{1}{x^2}\right)dx}{x^2\left(\frac{1}{x}-x\right)\sqrt{\frac{1}{x^2}+1+x^2}}$

$= -\int \frac{(1+1/x^2)dx}{(x-1/x)\sqrt{(x-1/x)^2+3}} = -\int \frac{dt}{t\sqrt{t^2+3}}$ (put $x - \frac{1}{x} = t$)

Again put $t^2 + 3 = s^2$ $\Rightarrow$ $2t\,dt = 2s\,ds = -\int \frac{s\,ds}{s(s^2-3)}$

$= -\int \frac{ds}{s^2-(\sqrt{3})^2} = -\frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{(x-1/x)^2+3}-\sqrt{3}}{\sqrt{(x-1/x)^2+3}+\sqrt{3}}\right| + c = -\frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{x^2+\frac{1}{x^2}+1}-\sqrt{3}}{\sqrt{x^2+\frac{1}{x^2}+1}+\sqrt{3}}\right| + c$

Ex.34 Evaluate $\int \frac{(x-1)dx}{(x+1)\sqrt{x^3+x^2+x}}$

Sol. Let, I = $\int \frac{(x-1)dx}{(x+1)\sqrt{x^3+x^2+x}} = \int \frac{(x^2-1)dx}{(x+1)^2\sqrt{x^3+x^2+x}} = \int \frac{x^2(1-1/x^2)dx}{(x^2+2x+1)\sqrt{x^3+x^2+x}}$

$= \int \frac{x^2\left(1-\frac{1}{x^2}\right)dx}{x^2\left(x+2+\frac{1}{x}\right)\sqrt{x+1+\frac{1}{x}}} = \int \frac{dt}{(t+2)\sqrt{t+1}}$ (put $x + \frac{1}{x} = t$, $(1 - 1/x^2)\,dx = dt$)

$= \int \frac{2z\,dz}{(z^2+1)z} = 2\int \frac{dz}{z^2+1} = 2\tan^{-1}(z) + c$ (put $t + 1 = z^2 \Rightarrow dt = 2zdz$)

$= 2\tan^{-1}(\sqrt{t+1}) + c = 2\tan^{-1}\sqrt{\frac{x^2+x+1}{x}} + c$

Ex.35 Evaluate $\int \frac{dx}{\sqrt{\{(x-\alpha)(\beta-x)\}}}$

Sol. Put $x = \alpha\cos^2\theta + \beta\sin^2\theta$ so that $dx = 2(\beta-\alpha)\sin\theta\cos\theta\,d\theta$
Also$(x-\alpha) = (\beta-\alpha)\sin^2\theta$, and $(\beta - x) = (\beta-\alpha)\cos^2\theta$

Making these substitutions,in the given integral = $\int \frac{2(\beta-\alpha)\sin\theta\cos\theta d\theta}{\sqrt{\{(\beta-\alpha)\cos^2\theta.(\beta-\alpha)\sin^2\theta\}}} = \int \frac{2(\beta-\alpha)\sin\theta\cos\theta}{(\beta-\alpha)\cos\theta\sin\theta}\,d\theta$

$= 2\int d\theta = 2\theta = \cos^{-1}(\cos 2\theta)$ ....(1)

But $x = \alpha\cos^2\theta + \beta\sin^2\theta$; $\therefore$ $2x = \alpha(1+\cos 2\theta) + \beta(1-\cos 2\theta)$
i.e., $(\beta-\alpha)\cos 2\theta = (\alpha+\beta-2x)$ or $\cos 2\theta = (\alpha+\beta-2x)/(\beta-\alpha)$

$\therefore$ from (1), we get the given integral = $\cos^{-1}\left(\frac{\alpha+\beta-2x}{\beta-\alpha}\right)$.

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Ex.36 Evaluate $I = \int \frac{dx}{(a+dx^2)\sqrt{b-ax^2}}$

Sol. Substituting $ax^2 = b\sin^2\theta \Rightarrow dx = \sqrt{\frac{b}{a}}\cos\theta\, d\theta$ $\therefore$ $I = \int \frac{\sqrt{\frac{b}{a}}\cos\theta d\theta}{\left(a+\frac{b^2}{a}\sin^2\theta\right)\sqrt{b-b\sin^2\theta}}$

$= \sqrt{a}\int \frac{\cos\theta d\theta}{(a^2+b^2\sin^2\theta).\cos\theta} = \sqrt{a}\int \frac{d\theta}{a^2+b^2\sin^2\theta}$, dividing N^r and D^r by $\cos^2\theta$. we get

$= \sqrt{a}\int \frac{\sec^2\theta d\theta}{a^2\sec^2\theta + b^2\tan^2\theta}$ put $\tan\theta = t$

$= \sqrt{a}\int \frac{dt}{a^2(1+t^2)+b^2t^2} = \frac{\sqrt{a}}{(a^2+b^2)}\int \frac{dt}{t^2+\frac{a^2}{a^2+b^2}} = \frac{1}{\sqrt{a(a^2+b^2)}}\tan^{-1}\left(\frac{t\sqrt{a^2+b^2}}{a}\right) + c$

$= \frac{1}{\sqrt{a(a^2+b^2)}} . \tan^{-1}\left(\frac{x\sqrt{a^2+b^2}}{a\sqrt{b-ax^2}}\right) + c$ (since, $t = \tan\theta = \frac{x}{\sqrt{b-ax^2}}$)

Ex.37 Integrate $1/(1 + 3\sin^2x)$.
Sol. Dividing Nr. and Dr. by $\cos^2 x$, we have

$I = \int \frac{dx}{1+3\sin^2 x} = \int \frac{\sec^2 x dx}{\sec^2 x + 3\tan^2 x} = \int \frac{\sec^2 x dx}{(1+\tan^2 x)+3\tan^2 x} = \int \frac{\sec^2 x dx}{1+\tan^2 x}$

Now putting $2\tan x = 5$ so that $2\sec^2x\, dx = dt$, we have $I = \frac{1}{2}\int \frac{dt}{1+t^2} = \frac{1}{2}\tan^{-1} t = \frac{1}{2}\tan^{-1}(2\tan x)$

E. INTEGRATION BY PARTS

$\int u.v dx = u\int v\,dx - \int\left[\frac{du}{dx}.\int v\,dx\right]dx$ where u & v are differentiable functions.

Note : While using integration by parts, choose u & v such that

(a) $\int v\,dx$ is simple & **(b)** $\int\left[\frac{du}{dx}.\int v\,dx\right]dx$ is simple to integrate.

This is generally obtained, by keeping the order of u & v as per the order of the letter in **ILATE**, where
I – Inverse function
L – Logarithmic function
A – Algebraic function
T – Trigonometric function
E – Exponential function

Ex.38 Integrate $x^n \log x$

Sol. We have $\int x^n \log x\, dx = \int (\log x).x^n dx$

$= (\log x) . \frac{x^{n+1}}{n+1} - \int \frac{1}{x}.\frac{x^{n+1}}{n+1}dx = (\log x) . \frac{x^{n+1}}{n+1} - \int \frac{x^n}{n+1}dx = (\log x) . \frac{x^{n+1}}{n+1} - \frac{x^{n+1}}{(n+1)^2} + C$

Ex.39 Evaluate $\int \frac{\log(\sec^{-1} x)dx}{x\sqrt{(x^2-1)}}$ dx

Sol. Put $\sec^{-1} x = t$ so that $\frac{1}{x\sqrt{(x^2-1)}}$ dx = dt.

Then the given integral $= \int \log\ t\ dt = \int (\log t).1\,dt = (\log t)\ .\ t - \int \frac{1}{t} t\ dt = t \log t - t + c$

$= t(\log t - \log e) + c = \sec^{-1} x (\log(\sec^{-1} x) - 1) + c = \sec^{-1} x \left(\log\left(\frac{\sec^{-1} x}{c}\right)\right) + c$

Ex.40 Evaluate $\int \tan^{-1}\sqrt{\left(\frac{1-x}{1+x}\right)}$ dx.

Sol. Put $x = \cos\theta$ so that $dx = -\sin\theta\ d\theta$. the given integral $= \int\left\{\tan^{-1}\sqrt{\left(\frac{1-\cos\theta}{1+\cos\theta}\right)}\right\}(-\sin\theta)d\theta$

$= -\int(\tan^{-1}(\tan\frac{\theta}{2})\sin\theta\,d\theta) = -\int\frac{\theta}{2}\sin\theta\ d\theta = -\frac{1}{2}\int\theta\sin\theta\ d\theta$

$= -\frac{1}{2}[\theta\ .\ (-\cos\theta) - \int(-\cos\theta)\,d\theta] = \frac{\theta\cos\theta}{2} - \frac{\sin\theta}{2} = \frac{1}{2}[x\cos^{-1} x - \sqrt{(1-x^2)}]$

Ex.41 Evaluate $\int x^2 \tan^{-1} x\,dx$.

Sol. We have $\int x^2\tan^{-1} x\,dx = \frac{x^3}{3}\tan^{-1} x - \int\frac{x^3}{3}.\frac{1}{1+x^2}$,

integrating by parts taking x^2 as the second function

$\frac{x^3}{3}\tan^{-1} x - \frac{1}{3}\int\frac{x(x^2+1)-x}{1+x^2}$ dx [$\because$ $x^3 = x(x^2+1) - x$]

$= \frac{x^3}{3}\tan^{-1} x - \frac{1}{3}\int x\,dx + \int\frac{1}{3}.\frac{1}{2}\int\frac{2x}{1+x^2}dx = \frac{x^3}{3}\tan^{-1} x - \frac{x^2}{6} + \frac{1}{6}\log(1+x^2) + c$

Ex.42 Evaluate $\int\sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right)$ dx.

Sol. $I = \int\sin^{-1}\left\{\frac{2x+2}{\sqrt{4x^2+8x+13}}\right\} dx = \int\sin^{-1}\left(\frac{2x+2}{\sqrt{(2x+2)^2+3^2}}\right) dx$

$= \int\sin^{-1}\left(\frac{3\tan\theta}{3\sec\theta}\right)\frac{3}{2}\sec^2\theta\ d\theta = \frac{3}{2}\int\theta\sec^2\theta d\theta$ (put, $2x + 2 = 3\tan\theta \Rightarrow 2\,dx = 3\sec^2\theta\ d\theta$)

$= \frac{3}{2}\ (\theta \tan\theta - \int \tan\theta d\theta\ ) = \frac{3}{2}\ \{\theta \tan\theta - \log (\sec \theta)\} + c$

$I = \frac{3}{2}\left\{\frac{2x+2}{3}\tan^{-1}\left(\frac{2x+2}{3}\right) - \log\left(\sqrt{1+\left(\frac{2x+2}{3}\right)^2}\right)\right\} + C$

$= \frac{3}{2}\left\{\frac{2}{3}(x+1)\tan^{-1}\left(\frac{2}{3}(x+1)\right) - \log\frac{\sqrt{4x^2+8x+13}}{3}\right\} + c$

$\Rightarrow\ \ I = (x + 1) \tan^{-1}\left(\frac{2}{3}(x+1)\right) - \frac{1}{4}\ \log (4x^2 + 8x + 13) + c$

Ex.43 If $\cos\theta > \sin\theta > 0$, then evaluate : $\int\left\{\log\left(\frac{1+\sin 2\theta}{1-\sin 2\theta}\right)^{\cos^2\theta} + \log\left(\frac{\cos 2\theta}{1+\sin 2\theta}\right)\right\}d\theta$

Sol. Here, $I = \int\left\{\log\left(\frac{1+\sin 2\theta}{1-\sin 2\theta}\right)^{\cos^2\theta} + \log\left(\frac{\cos 2\theta}{1+\sin 2\theta}\right)\right\}d\theta$

$= \int\left\{2\cos^2\theta \log\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right) - \log\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)\right\}d\theta = \int(2\cos^2\theta - 1)\log\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)d\theta$

$= \int \underset{\text{II}}{\cos 2\theta}.\underset{\text{I}}{\log\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)}d\theta$, applying integration by parts

$= \log\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right).\ \frac{\sin 2\theta}{2} - \int\frac{2}{\cos 2\theta}\ .\ \frac{\sin 2\theta}{2}d\theta = \frac{\sin 2\theta}{2}\ \log\left|\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right| + \frac{1}{2}\ \log |\cos 2\theta| + c$

REMEMBER THIS

(i) I $\int e^{ax}$. sinbx dx $= \frac{e^{ax}}{a^2+b^2}$ (a sin bx − b cos bx) + c

(ii) $\int e^{ax}$. cos bx dx $= \frac{e^{ax}}{a^2+b^2}$ (a cos bx + b sin bx) + c

Evaluate $I = \int e^{ax}\sin bx dx$

Integrating by parts taking sin bx as the second function,

We get $I = -\frac{e^{ax}\cos bx}{b} - \int ae^{ax}\left(-\frac{\cos bx}{b}\right)dx = -\frac{e^{ax}\cos bx}{b} + \frac{a}{b}\int e^{ax}\cos bx dx$

Again integrating by parts taking cos bx as the second function, we get

$I = -\frac{e^{ax}\cos bx}{b} + \frac{a}{b}\left[\frac{e^{ax}\sin bx}{b} - \int ae^{ax}\frac{\sin bx}{b}dx\right]$

or $I = -\dfrac{e^{ax}\cos bx}{b} + \dfrac{a}{b^2} e^{ax} \sin bx - \dfrac{a^2}{b^2}\int e^{ax}\sin bx dx$

or $I = \dfrac{e^{ax}}{b^2}$ (a sin bx − b cos bx) − $\dfrac{a^2}{b^2}$ I. $[\because \; e^{ax} \sin bx\ dx = 1]$

Transposing the term $-\dfrac{a^2}{b^2}$ I to the left hand side, we get $\left(1+\dfrac{a^2}{b^2}\right) I = \dfrac{e^{ax}}{b^2}$ (a sin bx − b cos bx)

or $\dfrac{1}{b^2}(a^2 + b^2) I = \dfrac{1}{b^2} e^{ax}$ (a sin bx − b cos bx) $\therefore I = \dfrac{e^{ax}}{a^2+b^2}$ (a sin bx − b cos bx)

Thus, $\int e^{2x}\sin x\, dx = \dfrac{e^{2x}}{5}(2\sin x - \cos x) + C$

Remark : **(i)** $\int e^x[f(x) + f'(x)]dx = e^x.f(x) + c$ **(ii)** $\int [f(x) + xf'(x)]dx = xf(x) + c$

Ex.44 Evaluate $\int \dfrac{xe^x}{(x+1)^2} dx$

Sol. We have $\int \dfrac{xe^x}{(x+1)^2}dx = \int xe^x \dfrac{1}{(x+1)^2}dx$

$\int \dfrac{xe^x}{(x+1)^2}dx = (xe^x)\left(-\dfrac{1}{x+1}\right) - \int (e^x + xe^x)\left(-\dfrac{1}{x+1}\right)dx$, [Note that the integral of $\dfrac{1}{(x+1)^2}$ is $-\dfrac{1}{x+1}$]

$= -\dfrac{xe^x}{(x+1)^2} + \int e^x(x+1)\dfrac{1}{x+1}dx = -\dfrac{xe^x}{x+1} + \int e^x dx = -\dfrac{xe^x}{x+1} + e^x + c$

$= e^x\left[1 - \dfrac{x}{x+1}\right] + c = e^x \dfrac{x+1-x}{x+1} + c = \dfrac{e^x}{x+1} + c$

Alternative solution

We have $\int \dfrac{xe^x}{(x+1)^2}dx = \int e^x \dfrac{(x+1)-1}{(x+1)}dx = \int e^x\left[\dfrac{1}{x+1} - \dfrac{1}{(x+1)^2}\right]dx$

$= \int e^x[f(x) + f'(x)dx$, where $f(x) = \dfrac{1}{x+1} = e^x \dfrac{1}{x+1} + c$

Ex.45 Evaluate $\int e^x \dfrac{2+\sin 2x}{1+\cos 2x}dx$

Sol. We have $\int e^x \dfrac{2+\sin 2x}{1+\cos 2x}dx = \int e^x\left[\dfrac{2}{2\cos^2 x} + \dfrac{2\sin x\cos x}{2\cos^2 x}\right]dx = \int e^x[\sec^2 x + \tan x]dx$

$= \int e^x[f(x) + f'(x)]dx$, where f(x) = tan x = e^x f(x) + c = e^x tanx + c

Ex.46 Evaluate $\int \frac{dx}{(x^2+a^2)^3}$ ....(i)

Sol. $I_1 = \int \frac{1}{(x^2+a^2)^2} dx$ ....(ii)

$= \int \underset{I}{\frac{1}{(x^2+a^2)^2}} . \underset{II}{1dx} = \frac{1}{(x^2+a^2)^2} . x - \int \frac{-2(2x)}{(x^2+a^2)^3} xdx = \frac{x}{(x^2+a^2)^2} + 4 \int \frac{x^2+a^2-a^2}{(x^2+a^2)^3} dx$

$= \frac{x}{(x^2+a^2)^2} + 4 \int \frac{1}{(x^2+a^2)^2} dx - 4a^2 \int \frac{dx}{(x^2+a^2)^3}$

$\Rightarrow I_1 = \frac{x}{(x^2+a^2)^2} + 4I_1 - 4a^2 . I$ (using (i) and (ii)) $\Rightarrow 4a^2 = \frac{x}{(x^2+a^2)^2} + \frac{3}{4a^2} I_1$ ....(iii)

{using, previous example, $I_1 = \int \frac{dx}{(x^2+a^2)^2} = \int \frac{x}{2a^2(x^2+a^2)^2} + \frac{1}{2a^3} \tan^{-1}\left(\frac{x}{a}\right) + c$}

$\Rightarrow I = \frac{x}{4a^2(x^2+a^2)^2} + \frac{3}{4a^2} \left\{\frac{x}{2a^2(x^2+a^2)} + \frac{1}{2a^3}\tan^{-1}\left(\frac{x}{a}\right)\right\} + C$

F. INTEGRATION BY REDUCTION FORMULAE

Ex.47 If $I_n = \int x^n\sqrt{a^2-x^2}\, dx$, prove that $I_n = -\frac{x^{n-1}(a^2-x^2)^{3/2}}{(n+2)} + \frac{(n+1)}{(n+2)} a^2 I_{n-2}$.

Sol. $I_n = \int x^n\sqrt{a^2-x^2}\, dx = \int x^{n-1}.\{x\sqrt{a^2-x^2}\}dx$

Applying integration by parts we get

$= x^{n-1} . \left\{\frac{(a^2-x^2)^{3/2}}{-3}\right\} + \int (n-1)x^{n-2} . \left\{-\frac{(a^2-x^2)^{3/2}}{3}\right\} dx$

$= -\frac{x^{n-1}(a^2-x^2)^{3/2}}{3} + \frac{(n+1)}{3} \int x^{n-2}.(a^2-x^2) \sqrt{a^2-x^2}\, dx$

$\Rightarrow I_n = -\frac{x^{n-1}(a^2-x^2)^{3/2}}{3} + \frac{(n-1)a^2}{3} I_{n-2} - \frac{(n-1)}{3} I_n$

$\Rightarrow I_n + \frac{(n-1)}{3} I_n = -\frac{x^{n-1}(a^2-x^2)^{3/2}}{3} + \frac{(n-1)a^2}{3} I_{n-2}$

$\Rightarrow \left(\frac{n+2}{3}\right) I_n = -\frac{x^{n-1}(a^2-x^2)^{3/2}}{3} + \frac{(n-1)a^2}{3} I_{n-2}$

$\Rightarrow I_n = \frac{x^{n-1}(a^2-x^2)^{3/2}}{(n+2)} + \frac{(n-1)a^2}{(n+2)} I_{n-2}$

Ex.48 Integration of $1/(x^2 + k)^n$.

Sol. Thus $\int \frac{1}{(x^2+k)^{n-1}}\ 1 \,.\, dx\ \frac{x}{(x^2+k)^{n-1}} - \int x.\frac{-(n-1)}{(x^2+k)^n}\ .\ 2x\ dx$

or $I_{n-1} = \frac{x}{(x^2+k)^{n-1}} + 2(n-1)\int \frac{(x^2+k)-k}{(x^3+k)^n}dx$, $\quad [\because\ x^2 = (x^2 + k) - k]$

or $I_{n-1} = \frac{x}{(x^2+k)^{n-1}} + 2\,(n-1)\left[\int \frac{dx}{(x^2+k)^{n-1}} - k\int \frac{dx}{(x^2+k)^n}\right]$

or $I_{n-1} = \frac{x}{(x^2+k)^{n-1}} + 2\,(n-1)\,I_{n-1} - 2k\,(n-1)\,I_n$. $\quad \therefore 2k\,(n-1)\,I_n = \frac{x}{(x^2+k)^{n-1}} + \{2(n-1) - 1\}\,I_{n-1}$

or $2k(n-1)\,I_n = \frac{x}{(x^2+k)^{n-1}} + (2n-3)\,I_{n-1}$.

Hence $\int \frac{dx}{(x^2+k)^{n-1}} = \frac{x}{2k(n-1)(x^2+k)^{n-1}} + \frac{(2n-3)}{2k(n-1)}\int \frac{dx}{(x^2+k)^{n-1}}$.

Above is the reduction formula for $\int [1/(x^2+k)^n]dx$. By repeated application of this formula the integral shall reduce to that of $\frac{1}{(x^2+k)}$ which is $\frac{1}{\sqrt{k}}\tan^{-1}\left(\frac{x}{\sqrt{k}}\right)$.

Ex.49 Integrate $1/(x^2 + 3)^3$.

Sol. By the reduction formula, we get

$\int \frac{dx}{(x^2+3)^3} = \frac{x}{12(x^2+3)^2} + \frac{3}{12}\int \frac{dx}{(x^2+3)^2}$, [putting n = 3 and k = 3 in the formula]

$= \frac{x}{12(x^2+3)^2} + \frac{1}{4}\left\{\frac{x}{6(x^2+3)} + \frac{1}{6}\int \frac{dx}{(x^2+3)}\right\}$,

(on applying the same reduction formula by putting n = 2 and k = 3)

$= \frac{x}{12(x^2+3)^2} + \frac{x}{24(x^2+3)} + \frac{1}{24\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} + c.$

Ex.50 Integrate $(x + 2) / (2x^2 + 4x + 3)^2$.

Sol. Here $(d/dx)\,(2x^2 + 4x + 3) = 4x + 4$.

$\therefore \int \frac{(x+2)dx}{(2x^2+4x+3)^2} = \int \frac{\frac{1}{4}(4x+4)+2-1}{2(x^2+4x+3)^2}dx = \frac{1}{4}\int \frac{(4+4)dx}{(2x^2+4x+3)^2} = \int \frac{(2-1)\,dx}{\left(x^2+2x+\frac{3}{2}\right)^2}$

$= \frac{1}{4}\int (2x^2+4x+3)^{-2}\,(4x+4)\,dx + \frac{1}{4}\int \frac{dx}{\left(x^2+2x+\frac{3}{2}\right)^2} = -\frac{1}{4(2x^2+4x+3)} + \frac{1}{4}\int \frac{dx}{\left\{(x+1)^2+\frac{1}{2}\right\}^2}$

Now put x + 1 = t and then applying the reduction formula, we get

$I = \frac{1}{4(2x^2+4x+3)} + \frac{1}{4}\left[\frac{(x+1)}{(x+1)^2+\frac{1}{2}} + \sqrt{2}\tan^{-1}\{\sqrt{2(x+1)}\}\right] + c$

Ex.51 Integrate $(2x + 3)/(x^2 + 2x + 3)^2$.

Sol. Here $(d/dx)\,(x^2 + 2x + 3) = 2x + 2$

$$\therefore \quad I = \int \frac{(2x+3)}{(x^2+2x+3)^2} = \int \frac{(2x+2+1)\,dx}{(x^2+2x+3)^2} = \int \frac{(2x+2)dx}{(x^2+2x+3)^2} = \int \frac{dx}{(x^2+2x+3)^2}$$

$$= -\frac{1}{(x^2+2x+3)} + \int \frac{dx}{(x^2+2x+3)^2} \qquad ....(i)$$

Now let $I_1 = \int \frac{dx}{[(x^2+1)^2+2]^2}$ (Put $x + 1 = \sqrt{2}$ tan t, so that $dx = \sqrt{2}\sec^2 t\,dt$)

$$\therefore \quad I_1 = \int \frac{\sqrt{2}\sec^2 t\,dt}{(2\tan^2 t+2)^2} = \frac{\sqrt{2}}{4}\int \cos^2 t\,dt = \frac{\sqrt{2}}{4}\int \frac{1}{2}\,(1 + \cos 2t)\,dt$$

$$= \frac{\sqrt{2}}{8}\left[+\frac{1}{2}\sin 2t\right] + \frac{\sqrt{2}}{8}\,[t + \sin t \cos t] + c$$

Now $\tan t = \frac{x+1}{\sqrt{2}}$. Therefore $\sin t = \frac{x+1}{\sqrt{\{(x+1)^2+2\}}} = \frac{x+1}{\sqrt{(x^2+2x+3)}}$, and $\cos t = \frac{\sqrt{2}}{\sqrt{(x^2+2x+3)}}$

Also $t = \tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)$. Hence $I_1 = \frac{\sqrt{2}}{8}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + \frac{\sqrt{2}}{8}\cdot\frac{x+1}{\sqrt{\{(x^2+2x+3)\}}}\cdot\frac{\sqrt{2}}{\sqrt{(x^2+2x+3)}}$.

$$= \frac{\sqrt{2}}{8}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + \frac{1}{4}\,\frac{x+1}{(x^2+2x+3)}$$

$$\therefore \quad I = -\frac{1}{x^2+2x+3} + \frac{1}{4}\,\frac{x+1}{x^2+2x+3} + \frac{\sqrt{2}}{8}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + c, \quad \text{from (i)}$$

$$= \frac{x+1-4}{4(x^2+2x+3)} + \frac{\sqrt{2}}{8}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + c = \frac{x-3}{4(x^2+2x+3)} + \frac{\sqrt{2}}{8}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + c$$

Ex.52 If $I_m = \int (\sin x + \cos x)^m\,dx$, then show that $mI_m = (\sin x + \cos x)^{m-1}.(\sin x - \cos x) + 2\,(m - 1)\,I_{m-2}$

Sol. $\because \quad I_m = \int (\sin x + \cos x)^m\,dx$

$= \int (\sin x + \cos x)^{m-1}.(\sin x + \cos x)\,dx$, applying integration by parts.

$= (\sin x + \cos x)^{m-1}(\cos x + \sin x) - \int (m-1)(\sin x + \cos x)^{m-2}\,dx\;.\;(\cos x - \sin x)\;.\;(\sin x - \cos x)\,dx$

$= (\sin x + \cos x)^{m-1}(\sin x - \cos x) + (m - 1)\int (\sin x - \cos x)^{m-2}(\sin x + \cos x)^2\,.\,dx$

As we know, $(\sin x + \cos x)^2 + (\sin x - \cos x)^2 = 2$,

$\therefore \quad I_m = (\sin x + \cos x)^{m-1}(\sin x - \cos x) + (m - 1)\int (\sin x + \cos x)^{m-2}\,\{2 - (\sin x + \cos x)^2\}\,dx$

$= (\sin x + \cos x)^{m-1}(\sin x - \cos x) + (m - 1)\int 2(\sin x + \cos x)^{m-2}\,dx - (m - 1)\int (\sin x + \cos x)^m dx$

$I_m = (\sin x + \cos x)^{m-1}(\sin x - \cos x) + (m - 1)I_{m-2} - (m - 1)I_m$

or $(m - 1)\,I_m + I_m = (\sin x + \cos x)^{m-1}(\sin x - \cos x) + 2\,(m - 1)\,I_{m-2}$

or $mI_m = (\sin x + \cos x)^{m-1}(\sin x - \cos x) + 2\,(m - 1)\,I_{m-2}$

Ex.53 If $I_{m,n} = \int \cos^m x.\cos nx.dx$, show that $(m + n)\, I_{m,n} = \cos^m x.\ \sin nx + m\, I_{(m-1,\ n-1)}$

Sol. We have, $I_{m,n} = \int \cos^m x.\cos nx\ dx$

$$= (\cos^m x)\left[\frac{\sin nx}{n}\right] \int m\cos^{m-1}(-\sin x).\ \frac{\sin nx}{n}\ dx = \frac{1}{n}\cos^m x\ .\ \sin nx + \frac{m}{n}\int \cos^{m-1} x(\sin x.\sin nx)\,dx$$

As we have $\cos(n - 1)x = \cos nx \cos x + \sin nx\ .\ \sin x$

$$\therefore\quad I_{m,n} = \frac{1}{n}\cos^m x\ .\ \sin x + \frac{m}{n}\int \cos^{m-1} x.\ \{\cos(n-1)x - \cos nx\ .\ \cos x\}\ dx$$

$$= \frac{1}{n}\cos^m x\ .\ \sin x + \frac{m}{n}\int \cos^{m-1} x.\cos(n-1)x.dx - \frac{m}{n}\int \cos^m x.\cos nx dx$$

$$= \frac{1}{n}\cos^m x\ .\ \sin nx + \frac{m}{n} I_{m-1,n-1} - \frac{m}{n} I_{m,n}$$

$$\Rightarrow\quad I_{m,n} + \frac{m}{n} I_{m,n} = \frac{1}{n}[\cos^m x\ .\ \sin nx + mI_{m-1,\ n-1}] \Rightarrow \left(\frac{m+n}{n}\right) I_{m,n} = \frac{1}{n}[\cos^m x\ .\ \sin nx + mI_{m-1,n-1}]$$

$$\Rightarrow\quad (m + n)\, I_{m,n} = \cos^m x\ .\ \sin nx + mI_{m-1,n-1}.$$

Ex.54 If I_n denotes $\int z^n e^{1/z}\, dz$, then show that $(n + 1)!\ I_n = I_0 + e^{1/z}(1!z^2 + 2!z^3 + \ldots + n!\ z^{n+1})$.

Sol. $I_n = \int z^n e^{1/z}\, dz$, applying integration by parts taking $e^{1/z}$ as first function and z^n as second function. We get,

$$I_n = \frac{e^{1/z}.z^{n+1}}{(n+1)} - \int e^{1/z}\left(-\frac{1}{z^2}\right).\frac{z^{n+1}}{n+1}dz = \frac{e^{1/z}.z^{n+1}}{(n+1)} + \frac{1}{(n+1)}\int e^{1/z}.z^{n-1}dz$$

$$\text{or}\quad I_n = \frac{e^{1/z}.z^{n+1}}{(n+1)} + \frac{I_{n-1}}{(n+1)} = \frac{e^{1/z}.z^{n+1}}{(n+1)} + \frac{I_{n-1}}{(n+1)}\left[\frac{e^{1/z}.z^n}{n} + \frac{1}{n}I_{1-2}\right]$$

$$= \frac{e^{1/z}(z)^{n+1}}{(n+1)} + \frac{e^{1/z}.(z)^n}{(n+1)n} + \frac{1}{(n+1)n} I_{n-2}$$

$$= \frac{e^{1/z}.(z)^{n+1}}{(n+1)} + \frac{e^{1/z}.(z)^n}{(n+1)n} + \frac{e^{1/z}.(z)^{n-1}}{(n+1)n.(n-1)} + \frac{1}{(n+1)n(n-1)} I_{n-3}$$

....
....

$$= \frac{e^{1/z}(z)^{n+1}}{(n+1)} + \frac{e^{1/z}.(z)^n}{(n+1)n} + \ldots + \frac{e^{1/z}.(z)^{n-1}}{(n+1)n\ldots3.2} + \frac{1}{(n+1)n(n-1)\ldots3.2} I_0$$

Multiplying both sides by $(n + 1)!$ We get,

$(n + 1)!\ I_n = (e^{1/z}\ .\ z^{n+1}\ .\ n! + e^{1/z}\ .\ z^2\ (n - 1)! + \ldots + e^{1/z}\ .\ z^3\ .\ (2)! + e^{1/z}\ .\ z^2\ .\ 1!) + I_0$

$\Rightarrow\quad I_n\,(n + 1)! = I_0 + e^{1/z}(1\, z^2 + 2!\ z^3 + \ldots + n!\ z^{n+1})$.

G. INTEGRATION OF RATIONAL FUNCTIONS USING PARTIAL FRACTIONS

In this section we show how to integer any rational function (a ratio of polynomials) by expressing it as a sum of simpler fractions, called partial fractions, that we already know how to integrate. To illustrate the method, observe that by taking the fractions 2/(x – 1) and 1/(x + 2) to a common denominator we obtain $\frac{2}{x-1} - \frac{1}{x+2} = \frac{2(x+2)-(x-1)}{(x-1)(x+2)} = \frac{x+5}{x^2+x-2}$

If we now reverse the procedure, we see how to integrate the function on the right side of this equation $\int \frac{x+5}{x^2+x-2}\,dx = \int\left(\frac{2}{x-1} - \frac{1}{x+2}\right)dx = -\,2\ell n\,|x-1| - \ln|x+2| + C$

To see how the method of partial fractions works in general, let's consider a rational function $f(x) = \frac{P(x)}{Q(x)}$

Where P and Q are polynomials. It's possible to express f as sum of simpler fractions provided that the degree of P is less than the degree of Q. Such a rational function is called proper. Recall that if

$P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$

where $a_n \neq 0$, then the degree of P is n and we write deg (P) = n.

If f is improper, that is, deg(P) $\geq$ deg (Q), then we must take the preliminary step of dividing Q into P (by division) until a remainder R(x) is obtained such that deg (R) < deg(Q). The division statement is

(1) $f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$ where S and R are also polynomials.

As the following example illustrates, sometimes this preliminary step is all that is required.

Ex.55 Evaluate $\int \frac{x^3+x}{x-1}\,dx$.

Sol. Since the degree of the numerator is greater than the degree of the denominator, we first perform the long division. This enables us to write

$$\int \frac{x^3+x}{x-1}\,dx \int\left(x^2+x+2+\frac{2}{x-1}\right)dx = \frac{x^3}{3} + \frac{x^2}{2} + 2x + 2\ell n|x-1| + C$$

The next step is to factor the denominator Q(x) as far as possible . It can be shown that any polynomial Q can be factored as a product of linear factors (of the form ax + b) and irreducible quadratic factors (of the form $ax^2 + bx + c$, where $b^2 - 4ac < 0$). For instance, if $Q(x) = x^4 - 16$, we could factor it as $Q(x) = (x^2 - 4)(x^2 + 4) = (x - 2)(x + 2)(x^2 + 4)$

The third step is to express the proper rational function R(x)/Q(x) (from equation 1) as a sum of partial fractions of the form $\frac{A}{(ax+b)^i}$ or $\frac{Ax+B}{(ax^2+bx+c)^j}$

A theorem in algebra guarantees that it is always possible to do this. We explain the details for the four cases that occur.

Case I : The Denominator Q(x) is a product of distinct linear factors.

This means that we can write $Q(x) = (a_1x + b_1)(a_2x + b_2) \ldots (a_kx + b_k)$

where no factor is repeated (and no factor is a constant multiple of another). In this case the partial fraction theorem states that there exist constants $A_1, A_2, \ldots, A_k$ such that.

(2) $$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \ldots + \frac{A_k}{a_kx+b_k}$$

These constants can be determined as in the following example.

Ex.56 Evaluate $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$.

Sol. Since the degree of the numerator is less than the degree of the denominator, we don't need to divide. We factor the denominator as $2x^3 + 3x^2 - 2x = x(2x^2 + 3x - 2) = x(2x - 1)(x + 2)$

Since the denominator has three distinct linear factors, the partial fraction decomposition of the integrand (2) has the form.

(3) $$\frac{x^2+2x-1}{x(2x-1)(x+2)} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}$$

To denominator the values of A, B and C, we multiply both sides of this equation by the product of the denominators, $x(2x - 1)(x + 2)$, obtaining.

(4) $x^2 + 2x - 1 = A(2x - 1)(x + 2) + Bx(x + 2) + Cx(2x - 1)$

Expanding the right side of equation 4 and writing it in the standard form for polynomials, we get

(5) $x^2 + 2x - 1 = (2A + B + 2C)x^2 + (3A + 2B - C)x - 2A$

The polynomials in Equation 5 are identical, so their coefficients must be equal. The coefficient of x^2 on the right side, $2A + B + 2C$, must equal the coefficient of x^2 on the left side-namely, 1. Likewise. The coefficients of x are equal and the constant terms are equal. This gives the following system of equation for A, B and C.

$2A + B + 2C = 1$ $3A + 2B - C = 2$ $-2A = -1$

Solving, we get $A = \frac{1}{2}$, $B = \frac{1}{5}$, and $C = -\frac{1}{10}$, and so

$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx = \int \left(\frac{1}{2}\frac{1}{x} + \frac{1}{5}\frac{1}{2x-1} - \frac{1}{10}\frac{1}{x+2}\right)dx = \frac{1}{2}\,\ell n\,|x| + \frac{1}{10}\,\ell n|2x-1| - \frac{1}{10}\,\ell n|x+2| + K$$

In integrating the middle term we have made the mental substitution $u = 2x - 1$, which gives $du = 2dx$ and $dx = du/2$.

Case II : Q(x) is a product of linear factors, some of which are repeated.

Suppose the first linear factor $(a_1x + b_1)$ is repeated r times, that is, $(a_1x + b_1)^r$ occurs in the factorization of Q(x). Then instead of the single term $A_1/(a_1x + b_1)$ in equation 2, we would use

(6) $$\frac{A_1}{a_1x+b_1} + \frac{A_2}{(a_1x+b_1)^2} + \ldots + \frac{A_r}{(a_1x+b_1)^r}$$

By way of illustration, we could write $\frac{x^3-x+1}{x^2(x-1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3}$

but we prefer to work out in detail a simpler example.

Ex.57 Evaluate $\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\, dx$

Sol. The first step is to divide. The result of long division is $\frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}$ = x + 1 + $\frac{4x}{x^3 - x^2 - x + 1}$

The second step is to factor the denominator Q(x) = $x^3 - x^2 - x + 1$. Since Q(1) = 0, we know that x – 1 is a factor and we obtain $x^3 - x^2 - x + 1 = (x - 1)(x^2 - 1) = (x - 1)(x - 1)(x + 1) = (x - 1)^2 (x + 1)$

Since the linear factor x – 1 occurs twice, the partial fraction decompositoin is

$$\frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

Multiplying by the least common denominator $(x - 1)^2 (x + 1)$, we get

(7) $4x = A(x - 1)(x + 1) + B(x + 1) + C(x - 1)^2 = (A + C) x^2 + (B - 2x) x + (-A + B + C)$

Now we equate coefficients : A + C = 0

$$B - 2C = 4$$

$$-A + B + C = 0$$

Solving, we obtain A = 1, B = 2, and C = – 1, so

$$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\, dx = \int \left[x + 1 + \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1} \right] dx$$

$$= \frac{x^2}{2} + x + \ell n\, |x - 1| - \frac{2}{x-1} - \ell n |x + 1| + K = \frac{x^2}{2} + x - \frac{2}{x-1} + \ell n \left| \frac{x-1}{x+1} \right| + K$$

Case III : Q(x) contains irreducible quadratic factors, none of which is repeated.

If Q(x) has the factor $ax^2 + bx + c$, where $b^2 - 4ac < 0$, then in addition to the partial fractions in equation 2 and 6, the expression or R(x)/Q(x) will have a term of the form.

(8) $\frac{Ax + B}{ax^2 + bx + c}$ where A and B are constants to be determined. For instance, the function given by

f(x) = $x/[(x - 2)(x^2 + 1)(x^2 + 4)]$ has a partial fraction decomposition of the form

$$\frac{x}{(x-2)(x^2+1)(x^2+4)} = \frac{A}{x-2} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{x^2 + 4}$$

The term given in (8) can be integrated by completing the square and using the formula.

(9) $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$

Ex.58 Evaluate $\int \frac{2x^2 - x + 4}{x^3 + 4x}\, dx$

Sol. Since $x^3 + 4x = x(x^2 + 4)$ can't be factored further, we write $\dfrac{2x^2 - x + 4}{x(x^2+4)} = \dfrac{A}{x} + \dfrac{Bx + c}{x^2 + 4}$

Multiplying by $x(x^2 + 4)$, we have $2x^2 - x + 4 = A(x^2 + 4) + (Bx + C)x = (A + B)x^2 + Cx + 4A$
Equating coefficients, we obtain $A + B = 2 \quad C = -1 \quad 4A = 4$

Thus $A = 1$, $B = 1$ and $C = -1$ and so $\displaystyle\int \frac{2x^2 - x + 4}{x^3 + 4x}\,dx = \int\left(\frac{1}{x} + \frac{x-1}{x^2+4}\right)dx$

In order to integrate the second term we split it into to parts $\displaystyle\int \frac{x-1}{x^2+4}\,dx = \int \frac{x}{x^2+4}\,dx - \int \frac{1}{x^2+4}\,dx$

We make the substitution $u = x^2 + 4$ in the first of these integrals so that $du = 2x\,dx$. We evaluate the second integral by means of Formula 9 with $a = 2$.

$$\int \frac{2x^2 - x + 4}{x(x^2+4)}dx = \int \frac{1}{x}dx + \int \frac{x}{x^2+4}dx - \int \frac{1}{x^2+4}dx = \ell n\,|x| + \frac{1}{2}\,\ell n\,(x^2 + 4) - \frac{1}{2}\tan^{-1}(x/2) + K$$

Case IV : Q(x) Contains A repeated irreducible quadratic factor.
If Q (x) has the factor $(ax^2 + bx + c)^r$, where $b^2 - 4ac < 0$, then instead of the single partial fraction (8), the sum

(10) $$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \dots + \frac{A_rx + B_r}{(ax^2 + bx + c)^r}$$

occurs in the partial fraction decomposition of R(x)/Q(x), each of the terms in (10) can be integrated by first completing the square.

Ex.59 Write out the form of the partial fraction decomposition of the function $\dfrac{x^3 + x^2 + 1}{x(x-1)(x^2+x+1)(x^2+1)^3}$

Sol. $$\frac{x^3 + x^2 + 1}{x(x-1)(x^2+x+1)(x^2+1)^3} = \frac{A}{x} + \frac{B}{x-1} + \frac{Cx + D}{x^2+x+1} + \frac{Ex + F}{x^2+1} + \frac{Gx + H}{(x^2+1)^2} + \frac{Ix + J}{(x^2+1)^3}$$

Ex.60 Evaluate $\displaystyle\int \frac{1 - x + 2x^2 - x^3}{x(x^2+1)^2}dx$

Sol. The form of the partial fraction decomposition is $\dfrac{1 - x + 2x^2 - x^3}{x(x^2+1)^2} = \dfrac{A}{x} + \dfrac{Bx + C}{x^2+1} + \dfrac{Dx + E}{(x^2+1)^2}$

Multiplying by $x(x^2 + 1)^2$, we have $-x^3 + 2x^2 - x + 1 = A(x^2 + 1)^2 + (Bx + C)x(x^2 + 1) + (Dx + E)x$
$= A(x^4 + 2x^2 + 1) + B(x^4 + x^2) + C(x^3 + x) + Dx^2 + Ex = (A + B)x^4 + Cx^3 + (2A + B + D)x^2 + (C + E)x + A$
If we equate coefficient, we get the system $A + B = 0 \quad C = -1 \quad 2A + B + D = 2 \quad C + E = -1 \quad A = 1$
Which the solution $A = 1$, $B = -1$, $D = 1$, and $E = 0$. Thus

$$\int \frac{1 - x + 2x^2 - x^3}{x(x^2+1)^2}dx = \int\left(\frac{1}{x} - \frac{x+1}{x^2+1} + \frac{x}{(x^2+1)^2}\right)dx = \int \frac{dx}{x} - \int \frac{x}{x^2+1}dx - \int \frac{dx}{x^2+1} + \int \frac{x\,dx}{(x^2+1)^2}$$

$= \ell n\ |x| - \frac{1}{2}\ \ell n\ (x^2 + 1) - \tan^{-1}x - \frac{1}{2(x^2+1)} + K$

We note that sometimes partial fractions can be avoided when integrating a rational function. For instance, although the integral $\int \frac{x^2+1}{x(x^2+3)}dx$

could be evaluated by the method of case III, it's much easier to observe that if $u = x(x^2 + 3) = x^3 + 3x$, then $du = (3x^2 + 3)\ dx$ and so $\int \frac{x^2+1}{x(x^2+3)}\ dx = \frac{1}{3}\ \ell n\ |x^2 + 3x| + C$

Ex.61 Evaluate $\int \frac{x^3-3x^2+2x-3}{(x^2+1)^2}\ dx$.

Sol. In this example there is a repeated quadratic polynomial in the denominator. Hence, according to our previous discussion $\frac{x^3-3x^2+2x-3}{(x^2+1)^2} = \frac{A_1x+B_1}{x^2+1} + \frac{A_2x+B_2}{(x^2+1)^2}$

For some constants A_1, B_1, A_2 and B_2

An easy way to determine these constant is as follows. By long division,

$\frac{x^3-3x^2+2x-3}{x^2+1} = x - 3 + \frac{x}{x^2+1}$ and therefore $\frac{x^3-3x^2+2x-3}{x^2+1} = \frac{x-3}{x^2+1} + \frac{x}{(x^2+1)^2}$

Thus $A_1 = 1$, $B_1 = -3$, $A_2 = 1$ and $B_2 = 0$

we know have $\int \frac{x^3-3x^2+2x-3}{(x^2+1)^2}\ dx = \int \frac{x}{x^2+1}\ dx - \int \frac{3}{x^2-1}\ dx + \int \frac{x}{(x^2+1)^2}\ dx$

$= \frac{1}{2}\ell n\ (x^2 + 1) - 3\tan^{-1} x - \frac{1}{2(x^2+1)} + C$

Ex.62 Evaluate $\int \frac{dx}{\cos x + \operatorname{cosec} x}$

Sol. $I = \int \frac{dx}{\cos x + \frac{1}{\sin x}} = \int \frac{\sin x dx}{\cos x.\sin x + 1} = \int \frac{2\sin x dx}{2 + 2\sin x\ \cos x}dx = \int \frac{2\sin x}{2+\sin 2x}dx$

$= \int \frac{[(\sin x + \cos x) + (\sin x - \cos x)]dx}{2+\sin 2x} = \int \frac{\sin x + \cos x}{2 + \sin 2x}\ dx + \int \frac{\sin x - \cos x}{2+\sin 2x}dx$

$= \int \frac{\sin x + \cos x}{3-(1-\sin 2x)}\ dx + \int \frac{\sin x - \cos x}{1+(1+\sin 2x)} = \int \frac{\sin x + \cos x}{3-(\sin x - \cos x)^2}\ .\ dx + \int \frac{\sin x - \cos x}{1+(\sin x + \cos x)^2}\ .\ dx$

put sin x − cos x = s and sin x + cosx = t $\Rightarrow$ (cosx + sinx) dx = ds and (cosx − sin x) dx = dt

$$I = \int \frac{ds}{3-s^2} - \int \frac{dt}{1+t^2} = \int \frac{ds}{(\sqrt{3})^2-(s)^2} - \int \frac{dt}{1+t^2} = \frac{1}{2\sqrt{3}} \log \left|\frac{\sqrt{3}+s}{\sqrt{3}-s}\right| - \tan^{-1} t + c$$

$$= \frac{1}{2\sqrt{3}} \log \left|\frac{\sqrt{3}+\sin x-\cos x}{\sqrt{3}-\sin x+\cos x}\right| - \tan^{-1} (\sin x + \cos x) + c$$

Ex.63 Evaluate $\int \frac{\tan^{-1} x}{x^4} dx$.

Sol. $I = \int \frac{\tan^{-1} x}{x^4} dx - \int \tan^{-1} x . \frac{1}{x^4} dx = (\tan^{-1} x)\left(-\frac{1}{3x^2}\right) - \int \frac{1}{1+x^2} . \frac{1}{(-3x^3)} dx$

$$= -\frac{\tan^{-1} x}{3x^3} + \frac{1}{3}\int \frac{dx}{x^3(1+x^2)} \quad \text{Put } 1 + x^2 = t \Rightarrow 2x\,dx = dt$$

$$= -\frac{\tan^{-1} x}{3x^3} + \frac{1}{6}\int \frac{dt}{(t-1)^2 . t} \qquad I = -\frac{\tan^{-1} x}{3x^3} + \frac{1}{6} I_1 \qquad ....(1)$$

where, $I_1 = \int \frac{1}{(1-t)^2 t} dt = \int \left(\frac{A}{t-1} + \frac{B}{(t-1)^2} + \frac{C}{t}\right) dt$

Comparing coefficients we get, A = −1, B = 1, C = 1

$\therefore \quad I_1 = \int \left\{-\frac{1}{(t-1)} + \frac{1}{(t-1)^2} + \frac{1}{t}\right\} dt = -\log|t-1| - \frac{1}{(t-1)} + \log|t| \qquad \therefore$ From (i) and (ii), we get

$$I = \frac{\tan^{-1} x}{3x^3} + \frac{1}{6}\left\{-\log|x^2| - \frac{1}{x^2}\log|1+x^2|\right\} + c \qquad I = -\frac{\tan^{-1} x}{3x^3} + \frac{1}{6}\log\left|\frac{x^2+1}{x^2}\right| - \frac{1}{6x^2} + c$$

Ex.64 Evaluate $\int \frac{1}{(e^x-1)^2} dx$.

Sol. We have $\int \frac{1}{(e^x-1)^2} dx + \int \frac{e^x}{e^x(e^x-1)^2} dx$, [multiplying the Nr. and Dr. by e^x]

$= \int \frac{dt}{t(t-1)^2}$, putting $e^x = t$ so that $e^x\,dx = dt$.

Now $\frac{1}{t(t-1)} \equiv \frac{A}{t} + \frac{B}{t-1} + \frac{C}{(t-1)^2} \Rightarrow 1 \equiv A(t-1)^2 + Bt(t-1) + Ct$ ....(1) (on resolving into partial fractions)

To find A, putting t = 0 on both sides of (1), we get A = 1.

To find C, put t = 1 and we get C = 1. Thus $1 \equiv (t-1)^2 + Bt(t-1) + t$

Comparing the coefficients of t^2 on both sides, we get

$0 = 1 + B$ or $B = -1$ $\quad\therefore\quad \frac{1}{t(t-1)^2} = \frac{1}{t} - \frac{1}{t-1} + \frac{1}{(t-1)^2}$

Hence $\int \frac{dt}{t(t-1)^2} = \int \frac{1}{t}dt - \int \frac{dt}{t-1} + \int \frac{dt}{(t-1)^2} = \log t - \log (t-1) - \{1/(t-1)\} + C$

$= \log e^x - \log (e^x - 1) - \{1/(e^x - 1)\} + c = x - \log (e^x - 1) - \{1/(e^x - 1)\} + c$

Ex.65 Integrate $(3x + 1) / \{(x - 1)^3 (x + 1)\}$.

Sol. Putting $x - 1 = y$ so that $x = 1 + y$, we get $\frac{3x+1}{(x-1)^3(x+1)} = \frac{3(1+y)+1}{y^3(2+y)} = \frac{4+3y}{y^3(2+y)}$

arranging the Nr. and the Dr. in ascending powers of y

$= \frac{1}{y^3}\left[2 + \frac{1}{2}y - \frac{1}{4}y^2 + \frac{1}{4}\frac{y^3}{2+y}\right]$, by actual division

$= \frac{2}{y^3} + \frac{1}{2y^2} - \frac{1}{4y} + \frac{1}{4}\cdot\frac{1}{(2+y)} = \frac{2}{(x-1)^3} + \frac{1}{2(x-1)^2} - \frac{1}{4(x-1)} + \frac{1}{4(x+1)}$

Hence the required integral of the given fraction $= \int\left[\frac{2}{(x-1)^3} + \frac{1}{2(x-1)^2} - \frac{1}{4(x-1)} + \frac{1}{4(x+1)}\right] dx$

$= \frac{-1}{(x-1)^2} - \frac{1}{2(x-1)} - \frac{1}{4}\log (x-1) + \frac{1}{4}\log (x+1) + c = \frac{-1}{(x-1)^2} - \frac{1}{2(x-1)} - \frac{1}{4}\log\frac{x+1}{x-1} + c$

Ex.66 Evaluate the integral $\int \frac{1}{x^3(x-1)}dx$.

Sol. Let $x = \sec^2\theta \Rightarrow dx = 2\sec^2\theta \tan\theta\, d\theta \Rightarrow I = \int \frac{2\sec^2\theta\tan\theta d\theta}{\sec^6\theta\tan\theta} \Rightarrow 2\int\cos^4\theta d\theta$

$I = 2\int\cos^4\theta d\theta = 2\int[(\cos^2 2\theta)^2]d\theta = 2\int\left[\frac{1+\cos 2\theta}{2}\right]^2 d\theta = \frac{2}{4}\int(\cos^2 2\theta + 2\cos 2\theta)\,d\theta$

$= \frac{1}{2}\left[\int d\theta + \int\cos^2 2\theta d\theta + 2\int\cos 2\theta d\theta\right] = \frac{1}{2}\left[\theta + \int\left(\frac{1+\cos 4\theta}{2}\right)d\theta + \frac{\sin 2\theta}{2}\right] + c$

$= \frac{12}{2}\left[\theta + \frac{\theta}{2} + \frac{\sin 4\theta}{8} + \sin 2\theta\right] + c = \frac{\theta}{2} + \frac{\theta}{4} + \frac{\sin 4\theta}{16} + \frac{\sin 2\theta}{2} + c = \frac{3\theta}{4} + \frac{\sin 2\theta}{2} + \frac{\sin 4\theta}{16} + c$ where $x = \sec^2\theta$

Ex.67 Integrate, $\int \frac{2e^{5x}+e^{4x}-4e^{3x}+4e^{2x}+2e^{x}}{(e^{2x}+4)(e^{2x}-1)^2}dx$.

Sol. Put $e^x = y \Rightarrow I = \int \frac{2y^4+y^3-4y^2+4y+2}{(y^2+4)(y^2-1)^2}dy = \int \frac{y(y^2+4)(y^4-2y^2+1)}{(y^2+4)(y^2-1)^2}dy = -\frac{1}{2(e^{2x}-1)} + \tan^{-1}\left(\frac{e^x}{2}\right) + c$

Ex.68 Integrate $\int \frac{dy}{y^2(1+y^2)^3}$.

Sol. Put $y = \tan\theta \quad \Rightarrow \quad \int \frac{dy}{y^2(1+y^2)^3} = \int \frac{\cos^6\theta}{\sin^2\theta}d\theta = \int \frac{(1-\sin^2\theta)^3\, d\theta}{\sin^2\theta}$

$= \int \frac{(1-3\sin^2\theta+3\sin^4\theta-\sin^6\theta)d\theta}{\sin^2\theta} = \int (\operatorname{cosec}^2\theta - 3 + 3\sin^2\theta - \sin^4\theta)\,d\theta$

$= -\frac{1}{y} - \frac{15}{8}\tan^{-1}y - \frac{1}{2}\sin(2\tan^{-1}y) - \frac{1}{32}\sin(4\tan^{-1}y) + c$

Ex.69 Evaluate $\int \frac{f(x)}{x^3-1}dx$, where f(x) is a polynomials of degree 2 in x such that f(0) = f(1) = 3 f(2) = –3.

Sol. Let , $f(x) = ax^2 + bx + c$ given, f(0) = f(1) = 3f(2) = –3

$\therefore$ f(0) = f(1) = 3f(2) = –3, f(0) = c = –3, f(1) = a + b + c = – 3, 3f(2) = 3 (4a + b + c) = – 3

on solving we get a = 1, b = – 1, c = –3 $\therefore f(x) = x^2 - x - 3 \Rightarrow I = \int \frac{f(x)}{x^3-1}dx = \int \frac{x^2-x-3}{(x-1)(x^2+x+1)}dx$

Using partical fractions, we get, $\frac{(x^2-x-3)}{(x-1)(x^2+x+1)} = \frac{A}{(x-1)} + \frac{Bx+C}{(x^2+x+1)}$

We get, A = –1, B = 2, C = 2

$\therefore \quad I = \int -\frac{1}{x-1}dx + \int \frac{(2x+2)}{(x^2+x+1)}dx = -\log|x-1| + \int \frac{(2x+2)}{(x^2+x+1)} + \int \frac{1-dx}{x^2+x+1}$

$= -\log|x-1| + \log|x^2+x+1| + \int \frac{dx}{(x+1/2)^2+(\sqrt{3}/2)^2} = \log|x-1| + \log|x^2+x+1| + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + c$

Ex.70 Integrate 1/(sinx + sin 2x).

Sol. We have $I = \int \frac{dx}{\sin x + \sin 2x} = \int \frac{dx}{\sin x + 2\sin x\cos x} = \int \frac{dx}{\sin x(1+2\cos x)}$

$= \int \frac{\sin x\, dx}{\sin^2(1+2\cos x)} = \int \frac{\sin x dx}{(1-\cos^2 x)(1+2\cos x)}$ Now putting cos x = t, so that – sin x dx = dt, we get

$I = -\int \frac{dt}{(1-t^2)(1+2t)} = -\int \frac{dt}{(1-t)(1+t)(1+2t)} = -\int \left[\frac{1}{6(1-t)} - \frac{1}{2(1-t)} + \frac{4}{3(1+2t)}\right]dt,$

$= \frac{1}{6}\log(1-t) + \frac{1}{2}\log(1+t) - \frac{2}{3}\log(1+2t) + c = \frac{1}{6}\log(1-\cos x) + \frac{1}{2}\log(1+\cos x) - \frac{2}{3}\log(1+2\cos x) + c$

H. INTEGRATION OF IRRATIONAL FUNCTIONS

Certain types of integrals of algebraic irrational expressions can be reduced to integrals of rational functions by a appropriate change of the variable. Such transformation of an integral is called its rationalization.

(i) If the integrand is a rational function of fractional powers of an independent variable x, i.e. the function $R\left(x, x^{\frac{p_1}{q_1}}, \ldots, x^{\frac{p_k}{q_k}}\right)$, then the integral can be rationalized by the substitution $x = t^m$, where m is the least common multiple of the numbers $q_1, q_2, \ldots, q_k$.

(ii) If the integrand is a rational function of x and fractional powers of a linear fractional function of the form $\frac{ax+b}{cx+d}$, then rationalization of the integral is effected by the substitution $\frac{ax+b}{cx+d} = t^m$ where m has the same sense as above.

Ex.71 Evaluate $\int \frac{dx}{\sqrt{(x+a)} + \sqrt{(x+b)}}$.

Sol. Rationalizing the denominator, we have $\int \frac{dx}{\sqrt{(x+a)} + \sqrt{(x+b)}} \int \frac{\sqrt{(x+b)} - \sqrt{(x+a)}}{(x+a)-(x+a)}\, dx$

$$= \int \frac{(x+b)^{1/2} - (x+a)^{1/2}}{b-a}\, dx = \frac{1}{b-a}\left[\frac{2}{3}(x+b)^{3/2} - \frac{2}{3}(x+a)^{3/2}\right] = \frac{2}{3}\,\frac{1}{(b-a)}\,[(x+b)^{3/2} - (x+a)^{3/2}] + c$$

Ex.72 Evaluate $I = \int \frac{x + \sqrt[3]{x^2} + \sqrt[6]{x}}{x(1+\sqrt[3]{x})}$.

Sol. The least common multiple of the numbers 3 and 6 is 6, therefore we make the substitution
$x = t^6$, $dx = 6t^5\, dt$.

whence $I = 6 \int \frac{(t^6 + t^4 + t)t^5}{t^6(1+t^2)}\, dt = 6 \int \frac{t^5 + t^3 + 1}{1+t^2} dt = 6 \int t^3 dt + 6 \int \frac{dt}{t^2+1} = \frac{3}{2} t^4 + 6 \text{ arc tan } t + C.$

Returning to x, we obtain $I = \frac{3}{2} x^{2/3} + 6 \text{ arc tan } \sqrt[6]{x} + C.$

Ex.73 Evaluate $I = \int \frac{(2x-3)^{1/2} dx}{(2x-3)^{1/3} + 1}$.

Sol. The integrand is a rational function of $\sqrt[6]{2x-3}$ therefore we put $2x - 3 = t^6$, whence
$dx = 3t^5\, dt$; $(2x - 3)^{1/2} = t^3$; $(2x - 3)^{1/3} = t^2$.

$I = \int \frac{3t^8}{t^2+1}\, dt = 3 \int (t^6 - t^4 + t^2 - 1)\, dt + 3 \int \frac{dt}{1+t^2} = 3\,\frac{t^7}{7} - 3\,\frac{t^5}{5} + 3\,\frac{t^3}{3} - 3t + 3 \text{ arc tan } t + C.$

Returning to x, we get

$$I = 3\left[\frac{1}{7}(2x-3)^{7/6} - \frac{1}{5}(2x-3)^{5/6}\,\frac{1}{3}(2x-3)^{1/2} - (2x-3)^{1/6} + \text{arc tan}(2x-3)^{1/6}\right] + C.$$

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Ex.74 Evaluate $\int \sqrt[3]{x}\ \sqrt[7]{1+\sqrt[3]{x^4}}\ dx$.

Sol. Let $x = t^3 \Rightarrow dx = 3t^2$ then $I = \int t(1+t^4)^{1/7} . 3t^2\,dt = 3\int t^3(1+t^4)^{1/7}dt$

Let $1 + t^4 = X^7 \Rightarrow 4t^3\,dt = 7X^6\,dX \quad = \frac{3}{4}.\int 7X^7 dX = \frac{21}{32} X^6 + C$. Therefore $I = \frac{21}{32}\ (1 + x^{4/3})^{8/7} + C$

Ex.75 Evaluate $I = \int \frac{2}{(2-x)^2}\ \sqrt[3]{\frac{2-x}{2+x}}\ dx$.

Sol. The integrand is a rational function of x and the expression $\sqrt[3]{\frac{2-x}{2+x}}$, therefore let us introduce the substitution $\sqrt[3]{\frac{2-x}{2+x}} = t;\ \frac{2-x}{2+x} = t^3$, Whence $x = \frac{2-2t^3}{1+t^3}$; $2 - x = \frac{4t^3}{1+t^3}$; $dx = \frac{-12t^2}{(1+t^3)^2}\ dt$.

Hence $I = -\int \frac{2(1+t^3)^2\ t\,.1.\,2t^2}{16t^6(1+t^3)^2}\ dt = -\frac{3}{2}\int \frac{dt}{t^3} = \frac{3}{4t^2} + C$. We get $I = \frac{3}{4}\sqrt[3]{\left(\frac{2+x}{2-x}\right)^2} + C$.

INTEGRAL OF THE TYPE $\int \frac{dx}{X\sqrt{Y}}$ WHERE X AND Y ARE LINEAR OR QUADRATIC EXPRESSION

Ex.76 Integrate $1/[(2x + 1)\ \sqrt{4x+3)}\ ]$.

Sol. Put $4x + 3 = t^2$, so that $4dx = 2tdt$ and $(2x + 1) = \frac{2(t^2-3)}{4} + 1 = \frac{t^2-3}{2} + 1 = \frac{t^2-1}{2}$

$$\int \frac{dx}{[(2x+1)\sqrt{(4x+3)}]} = \int \frac{\frac{1}{2}tdt}{\frac{1}{2}(t^2-1)t} = \int \frac{dt}{(t^2-1)} = \frac{1}{2}\log\left\{\frac{t-1}{t+1}\right\} = \frac{1}{2}\ \log\ \frac{\sqrt{(4x+3)}-1}{\sqrt{(4x+3)}+1} + c.$$

Ex.77 Evaluate $\int \frac{x^2dx}{(x-1)\sqrt{(x+2)}}$.

Sol. Put $(x + 2) = t^2$, so that $dx = 2t\ dt$, Also $x = t^2 - 2$.

$$\therefore \int \frac{x^2dx}{(x-1)\sqrt{(x+2)}} = \int \frac{(t^2-2)^2.2t\ dt}{(t^2-3)\,.t} = 2\int \frac{t^4-4t^2+4}{t^2-3}\ dt$$

$= 2\int [t^2-1) + \{1/(t^2-3)\}]\,dt$, dividing the numerator by the denominator

$= 2\ [\frac{1}{3}\ t^3 - t + \{1/(2\sqrt{3})\}\ \log\ \{(t - \sqrt{3})/(t + \sqrt{3})\}]$

$$= 2\left[\frac{(x+2)^{3/2}}{3} - \sqrt{(x+2)} + \frac{1}{2\sqrt{3}}\log\frac{\sqrt{(x+2)}-\sqrt{3}}{\sqrt{(x+2)}+\sqrt{3}}\right] + c.$$

Ex.78 Integrate $1/\{x^2\sqrt{(x+1)}\}$.

Sol. Put $(x + 1) = t^2$, so that $dx = 2t\,dt$. Also $x = t^2 - 1$.

$$\int \frac{dx}{x^2\sqrt{(x+1)}} = \int \frac{2t\,dt}{(t^2-1)^2 . t} = 2\int \frac{dt}{(t+1)^2(t-1)^2} .$$

$$= \int \frac{1}{2}\left[\frac{1}{(t+1)^2} + \frac{1}{(t+1)} + \frac{1}{(t-1)^2} - \frac{1}{(t-1)}\right] dt, \text{ by partial fractions}$$

$$= \frac{1}{2}\int \frac{dt}{(t+1)^2} + \frac{1}{2}\int \frac{dt}{(t+1)} + \frac{1}{2}\int \frac{dt}{(t-1)^2} - \frac{1}{2}\int \frac{dt}{(t-1)}$$

$\therefore \quad -\frac{1}{2}\{1/(t + 1)\} + \frac{1}{2}\log(t + 1) - \frac{1}{2}\{1/(t - 1)\} -$ log (t − 1) + c

= − {1/(t + 1)} + 1/(t − 1)] + log {(t + 1)/(t − 1)} + c = − + log + c

Ex.79 Integrate 1/[(1 + x)].

Sol. Put $(1 + x) = 1/t$, so that $dx = -(1/t^2)\,dx$.
Also $x = (1/t) - 1$.

$\therefore$ = = − = −

= − . (2dt = −) = − + c = − + c.

Ex.80 Evaluate .

Sol. Put $x = 1/t$, so that $dx = -(1/t^2)\,dt$.

$\therefore$ I = = = −

Now put $1 + t^2 = z^2$ so that $t\,dt = z\,dz$. Then

I = − = = .

= log = log + c $\quad [\because \; z^2 = 1 + t^2]$

= log + c = log + c $\quad [\because \; t = 1/x]$

Ex.81 Evaluate I = .

Sol. Here, I = I = = –

= =

∵ = = . dt –

Let, I = I_1 – I_2 ....(i)

Where I_1 = and I_2 = put (t – 1) = for I_1,

I_1 = = – = – log ....(ii)

For I_2, put (t + 1) = , I_2 = – = – log ....(iii)

∴ I = – + log + c

where, z = and S = .

Ex.82 Evaluate .

Sol. Let I = = – cot x ℓn (cosx –) +

= – cot x ℓn (cosx –)+ = – cot x ℓn (cosx –) + I_1+ c

Where I_1 = . = × dx

= dx = dx +

Put sin x = ⇒ cosx dx = – dt

= − cot x − x + c = − − cot x − x + c = − − cot x − x + c

$\Rightarrow$ I = − cot x ℓn (cos x − − − cot x − x + c

INTEGRATION OF A BINOMIAL DIFFERENTIAL

The integral dx, where m, n, p are rational numbers, is expressed through elementary functions only in the following three cases :

Case I : p is an integer. Then, if p > 0, the integrand is expanded by the formula of the binomial; but if p < 0, then we put $x = t^k$, where k is the common denominator of the fractions and n.

Case II : is an integer. We put $a + bx^n = t^{\alpha}$, where α is the denominator of the fraction p.

Case III : + p is an integer we put $a + bx^n = t^{\alpha}x^n$, where a is the denominator of the fraction p.

Ex.83 Evaluate I = dx.

Sol. I = . Here p = 2, i.e. an integer, hence we have case I.

I = dx = dx = + + + C.

Ex.84 Evaluate I = dx.

Sol. I = . Here m = − ; n = ; p = ; = = 1, i.e. an integer.

we have case II. Let us make the substitution. Hence, I = 6 .

Ex.85 Evaluate I = .

Sol. Here p = − is a fraction, = = − also a fraction, but + p = − − = − 3 is an integer, i.e. we have case III, we put $1 + x^4 = x^{4/2}$, Hence x = ; dx = −

Substituting these expression into the integral, we obtain

I = − = − = + − + C.

Returning to x, we get I = − + − + C

EULER'S SUBSTITUTIONS

Integrals of the form dx are calculated with the aid of the three Euler substitutions.

1. = t ± is a > 0;

2. = t ± if a > 0;

3. = $(x - \alpha)$ t if $ax^2 + bx + c = a(x - \alpha)(x - \beta)$

i.e. if α is real, I root of the trinomial $ax^2 + bx + c$.

Ex.86 Evaluate I = .

Sol. Here a = 1 > 0, therefore we make the substitution = t − x.

Squaring both sides of this equality and reducing the similar terms, we get $2x + 2tx = t^2 - 2$

whence x = ; dx = dt ⇒ 1 + = 1 + t − =

Substituting into the integral, we obtain I = dt =

Now let us expand the obtained proper rational fraction into partial fractions :

= + + .

Applying the method of undetermined coefficients we find : A = 1, B = 0, D = − 2.

Hence dt = − 2 = $\ell n\ |t + 1|$ + + C.

Returning to x, we get I = $\ell n\ (x + 1 +$) + + C.

Ex.87 Evaluate I = .

Sol. Since here c = 1 > 0, we can apply the second Euler substitution = tx − 1,

whence $(2t - 1)\,x = (t^2 - 1)\,x^2$; x = ;

dx = − 2 dt; x + =

Substituting into I, we obtain an integral of rational fraction :

= dt, = + + + .

By the method of undetermined coefficient we find A = 2; B = − ; D = − 3; E = −

Hence I = 2 − − 3 − =

= $2\ell n\ |t| -$ $\ell n|t - 1| +$ − $\ell n\ |t + 1|$ + C. where t = .

Ex.88 Evaluate I = .

Sol. In this case a < 0 and c < 0 therefore neither the first, nor the second, Euler substitution is applicable. But the quadratic trinomial $7x - 10 - x^2$ has real roots $\alpha = 2$, $\beta = 5$, therefore we use the third Euler substituion : = = (x – 2) t.

Whence $5 - x = (x - 2)\,t^2$; x = ; dx = – ; (x – 2) t = t = .

Hence I = – dt = – dt = – + C. Where t = .

Ex.89 Evaluate .

Sol. Let I = Put x + = t ....(i)

⇒ dx = dt ⇒ dx = dt ....(ii)

We know t = x + = x + ×

t = ⇒ t = x + and – = x – substracting we get,

2 = t + or = ...(iii)

from (i), (ii) and (iii) we get dx = dt

∴ I = = dt = + c

⇒ I = [x + $]^{n+1}$ + (x + $)^{n-1}$ + c

Ex.90 If y $(x - y)^2 = x$, then show that = ln $[(x - y)^2 - 1]$.

Sol. Let P = = $\ell n\ [(x - y)^2 - 1]$ $\Rightarrow$ = = ...(i)

Given y $(x - y)^2 = x$, differentiating both sides, we get = ...(ii)

$\therefore$ = = =

$\therefore$ = Which is true as given. $\therefore$ = log $\{(x - y)^2 - 1\}$

I. CAN WE INTEGRATE ALL CONTINUOUS FUNCTION ?

The questions arises : Will our strategy for integration enable us to find the integral of every continuous function ? For example, can we use it to evaluate dx ? The answer is no, at least not in terms of the functions that we are familiar with.

The functions that we have been dealing with in this books are called elementary functions. These are the polynomials, rational functions, power functions (x^3), exponential function (a^x), logarithmic functions trigonometric and inverse trigonometric functions, hyperbolic and inverse hyperbolic functions, and all functions that can be obtained from these by the five operations of addition, substraction multiplication, division, and composition for instance, the function f(x) = + ℓn (cosh x) − $xe^{\sin 2x}$ is an elementary function

If f is an elementary function, then f′ is an elementary function but dx need not be an elementary function. Consider f(x) = . Since f is continuous, its integral exists, and if we define the function F by F(x) = then we know from part 1 of the fundamental theorem of calculus that F′(x) =

Thus, f(x) = has an antiderivative F, but it has been proved that F is not an elementary function.

?This means that no matter how hard we try, we will never succeed in evaluating dx in term of the function we know. The same can be said of the following integrals.

dx